2022

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To cite this article: J A Manamtam *et al* 2022 *J. Phys.: Conf. Ser.* **2157** 012001

Sigma chromatic numbers of the middle graph of some families of graphs

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Abstract. Let G be a nontrivial connected graph and let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G, where adjacent vertices may have the same color. For a vertex v of G, the color sum $\sigma(v)$ of v is the sum of the colors of the vertices adjacent to v. The coloring c is said to be a sigma coloring of G if $\sigma(u) \neq \sigma(v)$ whenever u and v are adjacent vertices in G. The minimum number of colors that can be used in a sigma coloring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$. In this study, we investigate sigma coloring in relation to a unary graph operation called middle graph. We will show that the sigma chromatic number of the middle graph of any path, cycle, sunlet graph, tadpole graph, ladder graph, or triangular snake graph is 2 except for some small cases. We also determine the sigma chromatic number of the middle graph of stars.

1. Introduction

One of the research topics involving vertex coloring of graphs that has become of interest in recent years is the sigma coloring. Introduced by Chartrand, Okamoto, and Zhang [1], it is a type of vertex coloring that is neighbor-distinguishing; that is, it induces a proper vertex coloring.

Definition 1. *[1] For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same.*

(i) The color sum $\sigma(v)$ of v is the sum of the colors of the vertices in $N(v)$.

(ii) If $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices u and v of G, then c is called a sigma coloring of G.

(iii) The minimum number of colors required in a sigma coloring of a graph G is called the sigma chromatic number of G and is denoted by $\sigma(G)$.

Throughout this paper, we consider only graphs that are nontrivial, connected, simple, and undirected.

In [1], it was shown that $\sigma(G) \leq \chi(G)$ for any graph G. Sigma colorings have been studied with respect to different families of graphs and/or in relation to some graph operations. In the original work [1], the sigma chromatic number of any path, cycle, complete bipartite graph, or any complete multipartite graph was determined. The sigma chromatic number has been studied in different graphs such as circulant graphs [2], the join of a finite number of paths and cycles [3], and graph coronas involving complete graphs [4]. In this work, we will study sigma colorings in relation to the graph operation middle graph, which is defined as follows.

Definition 2. *[5] Let G be a graph and let $V'(G)$ be the set of all singletons $\{v\}$ each containing a vertex v of G. We define the middle graph $M(G)$ of G as the intersection graph $\Omega(F)$, where $F = V'(G) \cup E(G)$.*

Alternatively, the middle graph $M(G)$ of a graph G is a graph with vertex set $V(M(G)) = V(G) \cup E(G)$ and where distinct elements u, v in $V(M(G))$ are adjacent in $M(G)$ if and only if $u, v \in E(G)$ are adjacent in G or $u \in V(G)$ and $v \in E(G)$ are incident in G. As an example, the cycle C_5 is and its middle graph are shown in Figure 1.

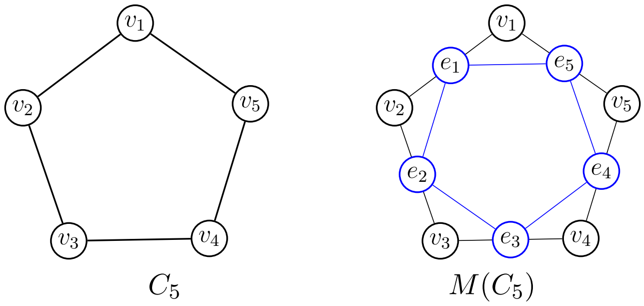

Figure 1. The cycle C_5 and its middle graph $M(C_5)$.

Since the introduction of the middle graph of a graph, the middle graph has been studied in relation to different graph colorings such as dominator coloring [6, 7], b-coloring [8], equitable coloring [9], r-dynamic coloring [10], and set coloring [11, 12]. In [13], it is shown that $\chi(M(G)) = \Delta(G) + 1$ for any graph G. The following is a known result on the sigma chromatic number.

Observation 3. *[1] Let G be a nontrivial connected graph. Then $\sigma(G) = 1$ if and only if every two adjacent vertices of G have different degrees.*

Given a set S of integers, we denote by $\sigma(S)$ the sum of the elements of S. We now present the following lemma, which will be useful in this study.

Lemma 4. *[1] For integers $k \geq 1$ and $N \geq 1$, let $\mathcal{U}_k = \{a_1, a_2, \ldots, a_k\}$ be a set of k positive integers such that $a_{i+1} \geq Na_i + 1$ for $1 \leq i \leq k-1$. Then for every two distinct multisets X and Y, both of cardinality at most N whose elements belong to $\mathcal{U}_k$, $\sigma(X) \neq \sigma(Y)$.*

In this study, we investigate sigma coloring in relation to a unary graph operation called middle graph. We will show that the sigma chromatic number of the middle graph of any path, cycle, sunlet graph, tadpole graph, ladder graph, or triangular snake graph is 2 except for some small cases. We also determine the sigma chromatic number of the middle graph of stars.

2. Sigma chromatic number of the middle graph of paths, cycles, sunlets, tadpoles, ladders, and triangular snakes

We begin by stating two observations. The first is a consequence of Lemma 4.

Observation 5. *Let G be a graph with $\Delta(G) = \Delta$ and let $d = \Delta + 1$. Let $A = \{1, d, d^2, \ldots, d^k\}$ for some $k \in \mathbb{Z}^+$. Let $c : V(G) \to A$ be a coloring of G. If u, v are adjacent vertices of G and $deg(u) \neq deg(v)$, then $\sigma(u) \neq \sigma(v)$.*

Proof. First, note that if we set $N = \Delta$, then for $1 \le i \le k-1$, $d^{i+1} = (\Delta+1)d^i = \Delta d^i + d^i \ge \Delta d^i + 1 = Nd^i + 1$.

Now, let X and Y be the multisets of colors of all the neighbors of u and v, respectively. Then $|X| \le N$ and $|Y| \le N$. Since $\deg(u) \ne \deg(v)$, we have $X \ne Y$ and Lemma 4 implies that $\sigma(u) = \sigma(X) \ne \sigma(Y) = \sigma(v)$. □

Observation 6. *Let $d > 2$ be an integer. For any $p, q, r, s \in \mathbb{Z}^+ \cup \{0\}$ such that $p \ne q$ and $r \ne s$,*

$$d^p - d^q = d^r - d^s \Leftrightarrow q = s \text{ and } p = r.$$

We now present the sigma chromatic number of the middle graph of paths.

Proposition 7. *Let P_n be a path graph of order $n \ge 2$. Then*

$$\sigma(M(P_n)) = \begin{cases} 1, & \text{if } n = 2, 4, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. The result can be easily verified for $n \le 4$ (see Figure 2, for example, for the case $n = 3$).

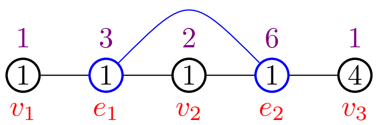

Figure 2. A sigma 2-coloring of $M(P_3)$.

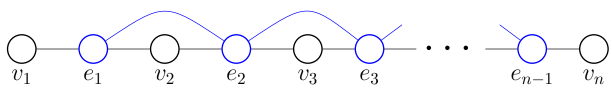

Figure 3. Vertex labels of $M(P_n)$.

Now let $n \ge 5$. Note that $\Delta = \Delta(M(P_n)) = 4$. Let $d = \Delta + 1 = 5$. We label the vertices of $M(P_n)$ as shown in Figure 3. Let c be a vertex coloring of $M(P_n)$ such that for $u \in V(M(P_n))$,

$$c(u) = \begin{cases} 5, & \text{if } u = v_j,\ 2 \le j \le n-1 \text{ and } j \equiv 0, 1 (\text{mod } 4) \\ 1, & \text{otherwise.} \end{cases}$$

We show that c is a sigma coloring of $M(P_n)$. Note that except for vertices that are of the form e_i for $2 \le i \le n-2$, the degrees of adjacent vertices in $M(P_n)$ are not equal. By Observation 5, we need only to show that $\sigma(e_i) \ne \sigma(e_{i+1})$ for all i, $2 \le i \le n-3$. Note that $N(e_i) = \{e_{i-1}, v_i, v_{i+1}, e_{i+1}\}$ and $c(e_i) = 1$ for all i, $2 \le i \le n-2$. We have the following cases:

Table 1. The $\sigma(e_i)$ and $\sigma(e_{i+1})$ on coloring of $M(P_n)$.

Case 1. If $i \equiv 0 (\text{mod } 4)$:	Case 2. If $i \equiv 1 (\text{mod } 4)$:
$c(v_i) = c(v_{i+1}) = 5$ $c(v_i) = c(v_{i+1}) = 1$ $c(e_{i-1}) = c(e_{i+1}) = 1$ $\Rightarrow \sigma(e_i) = 12$	$c(v_i) = 5$ $c(v_{i+1}) = 5$ $c(e_{i-1}) = c(e_{i+1}) = c(v_{i+1}) = 1$ $\Rightarrow \sigma(e_i) = 8$
$c(v_{i+1}) = 5$ $c(e_i) = c(e_{i+2}) = c(v_{i+2}) = 1$ $\Rightarrow \sigma(e_{i+1}) = 8$	$c(v_{i+1}) = c(v_{i+2}) = 1$ $c(e_i) = c(e_{i+2}) = 1$ $\Rightarrow \sigma(e_{i+1}) = 4$

Case 3. If $i \equiv 2 (\text{mod } 4)$:	Case 4. If $i \equiv 3 (\text{mod } 4)$:
$c(v_i) = c(v_{i+1}) = 1$ $c(e_{i-1}) = c(e_{i+1}) = 1$ $\Rightarrow \sigma(e_i) = 4$	$c(v_{i+1}) = 5$ $c(e_{i-1}) = c(e_{i+1}) = c(v_i) = 1$ $\Rightarrow \sigma(e_i) = 8$
$c(e_i) = c(e_{i+2}) = c(v_{i+1}) = 1$ $c(v_{i+2}) = 5$ $\Rightarrow \sigma(e_{i+1}) = 8$	$c(e_i) = c(e_{i+2}) = 1$ $c(v_{i+1}) = c(v_{i+2}) = 5$ $\Rightarrow \sigma(e_{i+1}) = 12$

This shows that $\sigma(e_i) \neq \sigma(e_{i+1})$ for $2 \leq i \leq n-3$. Thus for $n \geq 5$, c is a sigma coloring of $M(P_n)$ and $\sigma(M(P_n)) \leq 2$. On the other hand, $\sigma(M(P_n)) \geq 2$, by Observation 3. □

We now consider the middle graph of cycles.

Proposition 8. *Let C_n be a cycle graph of order $n \geq 3$. Then*

$$\sigma(M(C_n)) = 2.$$

Proof. Note that $\Delta = \Delta(M(C_n)) = 4$. Let $d = \Delta + 1 = 5$. We label the vertices of $M(C_n)$ for $n \geq 3$ as shown in Figure 4.

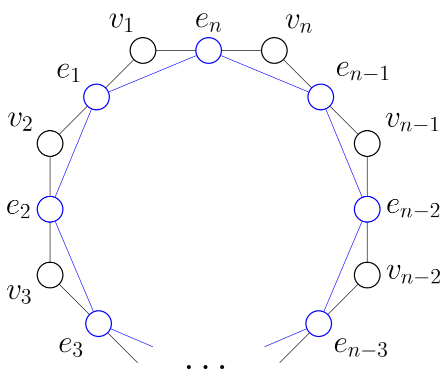

Figure 4. Vertex labels of $M(C_n)$.

Let $c : V(M(C_n)) \to \{1, 5\}$ be a vertex coloring of $M(C_n)$ defined as follows:

$$c(u) = \begin{cases} 5, & \text{if } u = v_1, \text{ or } u = v_2, \text{ or } u = e_i \text{ where } i \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that c is a sigma coloring of $M(C_3)$ (see Figure 5, with the color sums shown above each vertex).

We now show that c is a sigma coloring of $M(C_n)$ for $n \geq 4$. Note that except for those that are of the form e_i for $1 \leq i \leq n$, the degrees of adjacent vertices in $M(C_n)$ are not equal. By Observation 5, we only need to show that $\sigma(e_i) \neq \sigma(e_{i+1})$ for $1 \leq i \leq n-1$ and $\sigma(e_n) \neq \sigma(e_1)$. For $3 \leq i \leq n-1$, we have $N(e_i) = \{e_{i-1}, v_i, v_{i+1}, e_{i+1}\}$. Hence if i is odd, and $3 \leq i \leq n-2$, $\sigma(e_i) = 12 \neq 4 = \sigma(e_{i+1})$ and if i is even, $\sigma(e_i) = 4 \neq 12 = \sigma(e_{i+1})$. Lastly, for the case when $i = 1$ or 2, it is easy to verify that if n is odd, $\sigma(e_1) = 16$, $\sigma(e_2) = 8$, $\sigma(e_{n-1}) = 4$ and $\sigma(e_n) = 12$, whereas if n is even, $\sigma(e_1) = 20$, $\sigma(e_2) = 8$, $\sigma(e_{n-1}) = 12$ and $\sigma(e_n) = 8$. Thus, c is a sigma 2-coloring of $M(C_n)$. By Observation 3, it follows that $\sigma(M(C_n)) \neq 1$. Thus, $\sigma(M(C_n)) = 2$. □

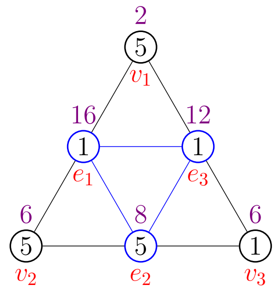

Figure 5. A sigma 2-coloring of $M(C_3)$.

We now present the sigma chromatic number of the middle graph of sunlet graphs. Recall that for $n \geq 3$, a sunlet graph is obtained by adding a pendant edge at each vertex of a cycle C_n.

Proposition 9. *Let S_n be a sunlet graph of order $2n$, where $n \geq 3$. Then*

$$\sigma\left(M(S_n)\right) = 2.$$

Proof. Note that $\Delta = \Delta(M(S_n)) = 6$. Let $d = \Delta + 1 = 7$. We label the vertices of $M(S_n)$ as shown in Figure 6.

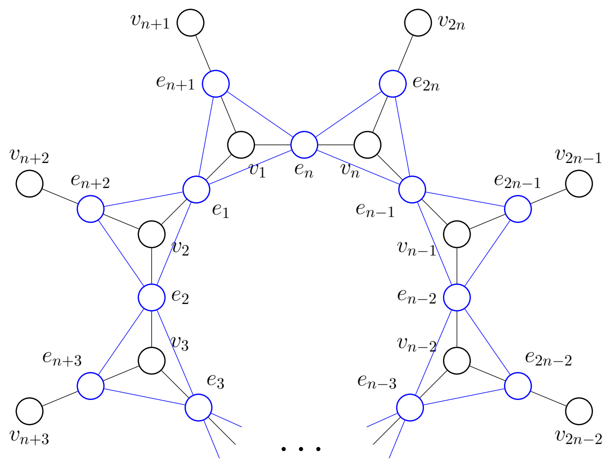

Figure 6. Vertex labels of $M(S_n)$.

Let $c : V\left(M(S_n)\right) \to \{1, 7\}$ be a vertex coloring of $M(S_n)$ defined as follows:

$$c(u) = \begin{cases} 7, & \text{if } u = v_1, \text{ or } u = v_2, \\ & \quad \text{or } u = e_i \text{ where } i \text{ is even and } 1 < i \leq n, \\ 1, & \text{otherwise.} \end{cases}$$

We show that c is a sigma coloring of $M(S_n)$. Note that except for vertices of the form e_i for $1 \leq i \leq n$. the degrees of adjacent vertices in $M(S_n)$ are not equal. By Observation 5, we only need to show that $\sigma(e_n) \neq \sigma(e_1)$ and that $\sigma(e_i) \neq \sigma(e_{i+1})$ for $i = 1, 2, 3, \ldots, n-1$. The proof is organized into two steps.

(i) For $2 \leq i \leq n-1$, note that $\sigma(e_i) = 18$ if i is odd and $\sigma(e_i) = 6$ if i is even. Thus, $\sigma(e_i) \neq \sigma(e_{i+1})$ for $2 \leq i \leq n-1$.

(ii) It can be shown that $\sigma(e_1) \neq \sigma(e_n)$, $\sigma(e_1) \neq \sigma(e_2)$, and $\sigma(e_n) \neq \sigma(e_{n-1})$ by considering two cases based on the parity of n.

Thus, c is a sigma coloring of $M(S_n)$ and $\sigma(M(S_n)) \leq 2$. For $1 \leq i \leq n-1$, e_i and e_{i+1} are adjacent and since $\deg e_i = \deg e_{i+1} = 6$, by Observation 3, we have $\sigma(M(S_n)) \neq 1$. Therefore, for $n \geq 3$, $\sigma(M(S_n)) = 2$. □

We now discuss the sigma chromatic number of the middle graph of tadpole graphs. Recall that a tadpole graph $T_{m,n}$ is obtained by joining a vertex of the cycle graph C_m, $m \geq 3$, to a pendant vertex of the path graph P_n, $n \geq 1$, using an edge.

Proposition 10. *Let $T_{m,n}$ be a tadpole graph of order $m+n$, where $m \geq 3$ and $n \geq 1$. Then*

$$\sigma(M(T_{m,n})) = 2.$$

Proof. Note that $\Delta = \Delta(M(T_{m,n})) = 5$. Let $d = \Delta + 1 = 6$. We label the vertices of $M(T_{m,n})$, where $m \geq 3$ and $n \geq 1$, as shown in Figure 7.

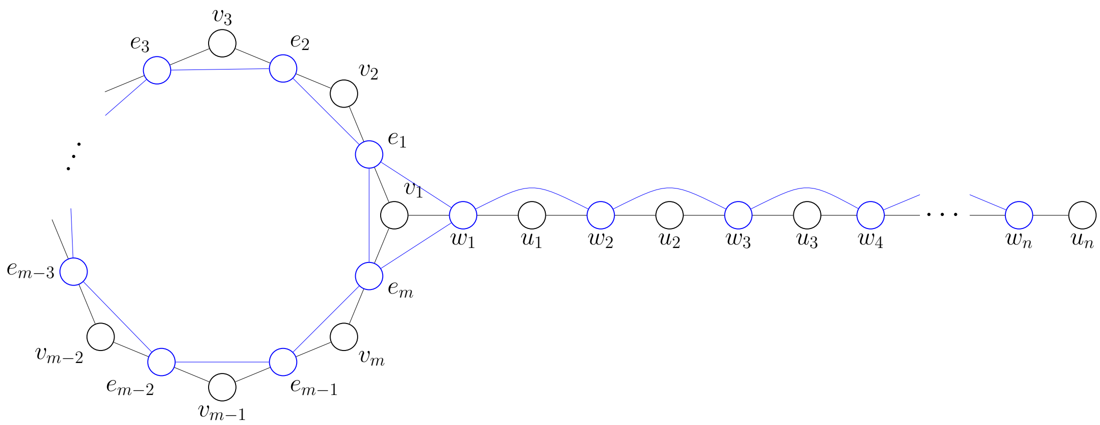

Figure 7. Vertex labels of $M(T_{m,n})$.

For $m \geq 3$, let $c : V(M(T_{m,n})) \to \{1, 6\}$ be a vertex coloring of $M(T_{m,n})$ define as follows:

$$c(u) = \begin{cases} 6, & \text{if } u = v_1, \text{ or } u = e_i \text{ where } i \text{ is even and } 2 \leq i \leq m-1, \\ & \text{or if } u = w_1, \text{ or } u = u_j \text{ where } j \equiv 0, 1 (\text{mod } 4) \text{ and } j \neq 1, \\ & \text{or if } u = e_m \text{ and } m \text{ is odd}, \\ 1, & \text{otherwise.} \end{cases}$$

We show that c is a sigma coloring of $M(T_{m,n})$. Note that the degrees of adjacent vertices in $M(T_{m,n})$ are not equal except: (i) for vertices that are of the form e_i and e_{i+1}, for $1 \leq i \leq m-2$; (ii) for distinct $u, v \in \{e_1, e_m, w_1\}$; and (iii) for vertices that are of the form w_j for $j = 2, 3, \ldots, n-1$ where $n \geq 4$. We now consider each of these cases.

(i) Suppose $m \geq 2$. For $2 \leq i \leq m-2$, $N(e_i) = \{v_1, v_{i+1}, e_{i-1}, e_{i+1}\}$. Hence, by definition of c, $\sigma(e_i) = 4$ if i is even and $\sigma(e_i) = 14$ if i is odd. Note that $\sigma(e_{m-1}) = 9$ whether m is odd or even. Thus, $\sigma(e_i) \neq \sigma(e_{i+1})$.

(ii) Suppose $m \geq 3$ and let $u, v \in \{e_1, e_m, w_1\}$. First, suppose $n \geq 2$. Then if m is even, $\sigma(e_1) = 20$, $\sigma(e_m) = 15$ and $\sigma(w_1) = 10$. If m is odd, $\sigma(e_1) = 25$, $\sigma(e_m) = 20$ and $\sigma(w_1) = 15$. Now, if $n = 1$, it is easy to verify that $\sigma(e_1) \neq \sigma(e_m)$, and since $\deg(w_1) = 4$ while $\deg(e_1) = \deg(e_m) = 5$, then $\sigma(w_1) \neq \sigma(e_1)$ and $\sigma(w_1) \neq \sigma(e_m)$. Thus, $\sigma(u) \neq \sigma(v)$.

(iii) Suppose $n \geq 4$. For $2 \leq j \leq n-2$, $N(w_j) = \{w_{j-1}, u_{j-1}, u_j, w_{j+1}\}$. It is easy to verify that either $\sigma(w_j) = 9$, $\sigma(w_j) = 14$, $\sigma(w_j) = 9$ and $\sigma(w_j)4$ when $j \equiv 0 (\text{mod } 4)$, $j \equiv 1 (\text{mod } 4)$, $j \equiv 2 (\text{mod } 4)$ or $j \equiv 3 (\text{mod } 4)$ respectively. Thus, $\sigma(w_i) \neq \sigma(w_{i+1})$ for $2 \leq j \leq n-2$.

Thus c is a sigma coloring of $M(T_{m,n})$. By Observation 3, $\sigma\left(M(T_{m,n})\right) \neq 1$. This proves the proposition. □

We now consider the middle graph of ladder graphs. Given path graphs $v_1 v_2 \cdots v_n$ and $u_1 u_2 \cdots u_n$, recall that a ladder graph is obtained by joining each v_i to u_i, for $1 \leq i \leq n$, with an edge.

Proposition 11. *Let L_n be a ladder graph of order $2n$, where $n \geq 2$. Then*

$$\sigma(M(L_n)) = 2.$$

Proof. Note that $\Delta = \Delta\left(M(L_n)\right) = 6$. Let $d = \Delta + 1 = 7$. We label the vertices of $M(L_n)$, where $n \geq 2$, as shown in Figure 8.

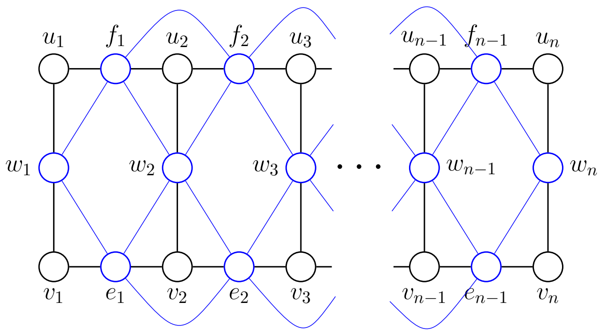

Figure 8. Vertex labels of $M(L_n)$.

By Observation 3, we have $\sigma(M(L_n)) \geq 2$ since there are at least two adjacent vertices with the same degree. To show that $\sigma(M(L_n)) \leq 2$, we define a vertex coloring $c : V(M(L_n)) \rightarrow \{1, 7\}$ as follows:

$$c(x) = \begin{cases} 1, & \text{if } x \in \{w_i, f_i, e_i\}, \text{ where } i \text{ is odd and } 1 \leq i \leq n-1, \\ & \text{or if } x = w_i, \text{ where } i \text{ is odd and } 1 \leq i \leq n, \\ 7, & \text{otherwise.} \end{cases}$$

Followng a similar procedure as in the proof of the previous proposition, it can be verified that $\sigma(u) \neq \sigma(v)$ for any two adjacent vertices u and v in $M(L_n)$. Thus, $\sigma(M(L_n)) = 2$. □

Recall that a triangular snake graph T_n of order $2n+1$ is obtained by adding new vertices $u_1, u_2, \ldots, u_n$ to a path graph $v_1 v_2 \cdots v_n v_{n+1}$ and joining v_i and v_{i+1} to the vertex u_i for $1 \leq i \leq n$. As an example, the triangular snake graph T_4 is shown in Figure 9.

Proposition 12. *Let T_n be a triangular snake graph of order $2n+1$, where $n \geq 1$, then*

$$\sigma(M(T_n)) = 2.$$

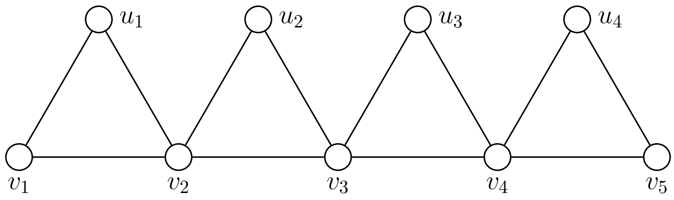

Figure 9. The triangular snake graph T_4.

Proof. For $n = 1$, $M(T_1) \cong M(C_3)$; thus, $\sigma(M(T_1)) = 2$. Moreover, it is easy to show that $\sigma(M(T_n)) = 2$ for $n = 2$. Suppose $n \geq 3$. Note that $\Delta = \Delta(M(T_n)) = 8$. Let $d = \Delta + 1 = 9$. We label the vertices of $M(T_n)$ as shown in Figure 10.

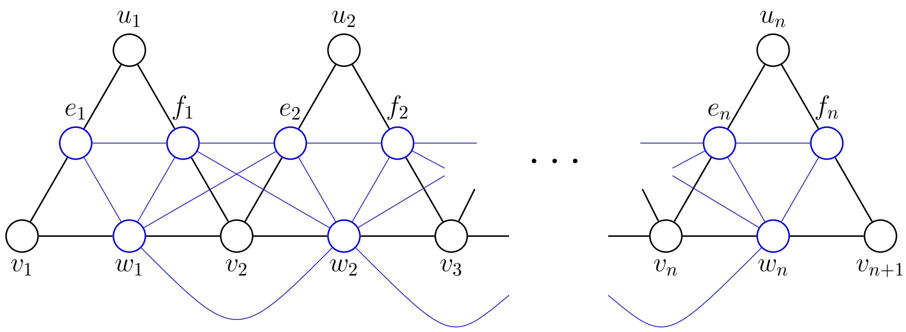

Figure 10. Vertex labels of $M(T_n)$.

By Observation 3, we have $\sigma(M(T_n)) \geq 2$. To show that $\sigma(M(T_n)) \leq 2$, we construct a sigma coloring $c : V(M(T_n)) \to \{1, 9\}$ that uses 2 colors. For $x \in V(M(T_n))$, we define $c(x)$ as follows:

$$c(x) = \begin{cases} 9, & \text{if } x = v_1 \text{ or} \\ & \text{if } x = e_i, 1 \leq i \leq n, \text{ or} \\ & \text{if } x = w_i, i \equiv 0 (\text{mod } 4), \text{ or} \\ & \text{if } x = v_{n+1}, n \equiv 0 (\text{mod } 4), \\ 1, & \text{otherwise.} \end{cases}$$

It can be easily verified that c is a sigma coloring of $M(T_n)$, following a similar argument as in the proofs of previous propositions. □

3. The sigma chromatic number of star graphs

In the previous section, we showed that the sigma chromatic number of the middle of paths, cycles, sunlet, tadpole and triangular snake graphs is 2 except for some small cases. In this section, we will show that the middle graph of a star graph $K_{1,k}$ is a function of k.

Observation 13. *Let $t \in \mathbb{Z}^+$. If $k = \left\lceil \frac{\sqrt{4t-3}+1}{2} \right\rceil$, then $t \leq k(k-1) + 1$.*

Proof. If $k = \left\lceil \frac{\sqrt{4t-3}+1}{2} \right\rceil$, then $\frac{\sqrt{4t-3}+1}{2} \leq k$. This inequality leads to $t \leq k(k-1) + 1$. □

Theorem 14. *Let k be a positive integer. Then*

$$\sigma(M(K_{1,k})) = \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil.$$

Proof. We label the vertices of $M(K_{1,k})$ as shown in Figure 11.

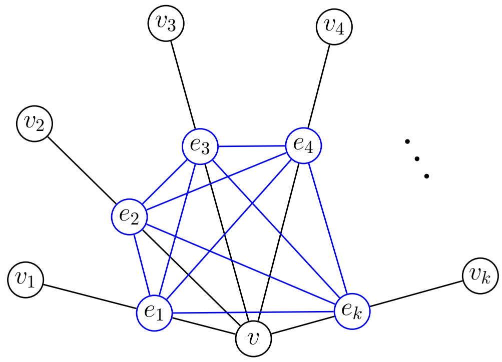

Figure 11. Vertex labels of $M(K_{1,k})$.

Let $p = \left\lceil \frac{\sqrt{4t-3}+1}{2} \right\rceil$. Now, we construct a sigma coloring of $M(K_{1,k})$ that uses p colors. Let $\Delta(M(K_{1,k})) = \Delta$ and set $d = \Delta + 1$ and $A = \{1, d, d^2, d^3, \ldots, d^{p-1}\}$. Moreover, set $B = \{(x_1, y_1) = (1, 1)\} \cup \{(x, y) : x, y \in A \text{ and } x \neq y\}$. We enumerate the elements of B as $(x_1, y_1), (x_2, y_2), \ldots, (x_{k(k-1)}, y_{k(k-1)}), (x_{k(k-1)+1}, y_{k(k-1)+1})$.

Define a coloring $c : V(M(K_{1,k})) \to A$ as follows: For $w \in V(M(K_{1,k}))$,

$$c(w) = \begin{cases} x_i, & \text{if } w = e_i, \text{ for } 1 \leq i \leq k, \\ y_i, & \text{if } w = v_i, \text{ for } 1 \leq i \leq k, \\ 1, & \text{if } w = v. \end{cases}$$

Since $p \geq \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil$, Observation 13 implies that $k \leq p(p-1) + 1 = |B|$. Thus, there are enough colors to construct c. To show that c is a sigma coloring, in light of Observation 5, we only need to show that $\sigma(e_i) \neq \sigma(e_j)$ for $1 \leq i < j \leq k$.

Set $S = \sum_{i=1}^{n} c(v_i)$. For $1 \leq i < j \leq t \leq n$,

$$\begin{aligned} \sigma(e_i) \neq \sigma(e_j) &\Leftrightarrow S - c(e_i) + c(v_i) \neq S - c(e_j) + c(v_j) \\ &\Leftrightarrow c(v_i) - c(e_i) \neq c(v_j) - c(e_j) \\ &\Leftrightarrow y_i - x_i \neq y_j - x_j. \end{aligned}$$

By the choice of colors, the construction of B and Observation 6, we have $y_i - x_i \neq y_j - x_j$ for $1 \leq i < j \leq k$. Hence $\sigma(e_i) \neq \sigma(e_j)$, for $1 \leq i, j \leq k$. Thus, c is a sigma coloring and $\sigma(M(K_{1,k})) \leq p = \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil$.

Next, we show that $\sigma(M(K_{1,k})) \geq \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil$. Let c be a sigma coloring of $M(K_{1,k})$, $k \geq 2$, that uses t colors. We have $\sigma(e_i) \neq \sigma(e_j)$, hence, $c(v_i) - c(e_i) \neq c(v_j) - c(e_j)$, for all $i \neq j$, $1 \leq i, j \leq k$. Thus, we must have k distinct differences $c(v_i) - c(e_i)$. Since c uses t colors, the number of distinct differences is at most $t(t-1) + 1$, where $t(t-1)$ is the number of ways to choose two distinct colors x and y, and additional 1 for the 0 difference, $0 = x - x$ for any color x. But then, $t(t-1) + 1 \geq k$ implies that $t \geq \frac{\sqrt{4k-3}+1}{2}$. Thus, $t \geq \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil$ since t must be an integer.

Therefore, for $k \geq 2$, $\sigma(M(K_{1,k})) = \left\lceil \frac{\sqrt{4k-3}+1}{2} \right\rceil$. □

To illustrate, we give a sigma coloring of $M(K_{1,5})$ in Figure 12 below, using the colors 1, 7 and 7^2.

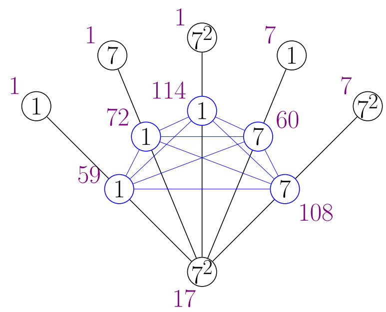

Figure 12. A sigma coloring of $M(K_{1,5})$.

4. Conclusion

In this study, we investigate sigma coloring in relation to a unary graph operation called middle graph. We showed that the sigma chromatic number of the middle graph of any path, cycle, sunlet graph, tadpole graph, ladder graph, or triangular snake graph is 2 except for some small cases. We also determine the sigma chromatic number of the middle graph of stars. For further investigations, other researchers may wish to pursue the following topics:

(i) Determine the sigma chromatic number of the middle graph of binary trees of height 2.

(ii) Determine the sigma chromatic number of the middle graph of families of graphs not covered in this paper such as bistars, etc.

Acknowledgement

The authors would like to thank the Office of the Vice President for the Loyola Schools of Ateneo de Manila University. We would also like to thank the organizers of the ICCGANT 2021 for the friendly welcome and hosting a very enlightening conference.

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