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## The N-integral

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To the paper

### *The N-integral*

The authors introduce a new version of the Henstock-Kurzweil (HK) type integral (called N-integral), which enables to avoid tags from some fixed "small" set. More precisely,

$$(N) \int_a^b f = A \quad \text{iff}$$

there is a null set  $S \subset [a, b]$  such that for each  $\varepsilon > 0$  there is a gauge  $\delta$  such that

$$\left| \sum_{\xi_j \notin S} f(\xi_j) [\alpha_j - \alpha_{j-1}] - A \right| < \varepsilon$$

holds for each  $\delta$ -fine tagged partition  $\{(\xi_j, [\alpha_{j-1}, \alpha_j])\}$  of  $[a, b]$ .

They establish the basic theory for such an integral, like Cauchy criterion, linearity, comparison with the HK integral, lemma of the Saks-Henstock type. Main results are presented in sections 3–5. Section 3 characterizes the class of N-integrable functions. Nice result is Theorem 3.5 which says a.o. that the function  $f$  is N-integrable iff it is HK integrable and continuous almost everywhere. Next section deals with the properties of the avoided sets and, as a result, the authors show that the N-integral covers the improper Riemann integrals. Finally, the last section provides several crucial examples and further interesting results, like Theorem 5.2 which says that every BVG\* function is almost everywhere continuous. To my opinion, the paper provides a considerable contribution to the integration theory, results are new and original and the proofs are correct. Therefore, **I recommend its publication in Journal of Indonesian Mathematica Society.**

To the presentation I have only a few formal comments that are listed below. Finally, let me point out that there is an interesting connection between the notion of the N-integral introduced by the submitted paper and the KN-integral introduced by P. Krejčí in his paper

The Kurzweil integral with exclusion of negligible sets.

*Mathematica Bohemica* 128 (3), 2003, 277-292.

He defined this Kurzweil-Stieltjes type integral as follows:

$$(KN) \int_a^b f dg = A \quad \text{iff}$$

for each  $\varepsilon > 0$  there are a gauge  $\delta$  and a negligible set  $S$  such that

$$\left| \sum f(\xi_j) [g(\alpha_j) - g(\alpha_{j-1})] - A \right| < \varepsilon$$

holds for each  $\delta$ -fine tagged partition  $\{(\xi_j, [\alpha_{j-1}, \alpha_j])\}$  of  $[a, b]$  such that  $\alpha_j \notin S$ .

It's remarkable that, while the KN-integral is a generalization of the Kurzweil-Stieltjes integral, in case of the N-integral just a change of the order of quantifiers and moving control from division points to tags leads to the less general integral, which, on the other hand, is a very nice characterization of functions continuous almost everywhere (or, in other words, of BVG\* functions).

### Further comments

- page 1, lines 6-7 On the 3rd line of Abstract, insert "of" between "set" and "N-integrable", and on the 7th line insert "assertions" between "following" and "are equivalent". Further, it seems to me that it would be better to emphasize in Abstract just the equivalence between (1) and (3) and to leave out the assertion (2) here.
- page 1, line 33 As the first definition of the gauge integral belongs to Kurzweil, it would be fair to speak here and throughout the paper about the Henstock-Kurzweil integral as it is now generally accepted, or shortly about the HK-integral.
- page 3, line 9 Replace "division" by "divisions".
- page 5, line 21 Replace "subet" by "subset".
- page 11, line 16 Insert "zero" between "measure" and "with".

## THE $N$ -INTEGRAL

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**Abstract.** In this paper, we introduced a Henstock-type integral named  $N$ -integral of a real valued function  $f$  on a closed and bounded interval  $[a, b]$ . The set  $N$ -integrable functions lie entirely between Riemann integrable functions and Henstock-Kurzweil integrable functions. Furthermore, this new integral integrates all improper Riemann integrable functions even if they are not Lebesgue integrable. It was shown that for a Henstock-Kurzweil integrable function  $f$  on  $[a, b]$ , the following are equivalent:

- (1) The function  $f$  is  $N$ -integrable;
- (2) There exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v - u| < \epsilon$$

where  $\phi_S(D) = \{(\xi, [u, v]) \in D : \xi \notin S\}$  and

$$\Gamma_\epsilon = \{(\xi, [u, v]) : |f(v) - f(u)| \geq \epsilon\}$$

and

- (3) The function  $f$  is continuous almost everywhere.

A characterization of continuous almost everywhere functions was also given.

*Key words and Phrases:*  $N$ -integral, Continuity almost everywhere, Henstock-Kurzweil integral.

### 1. INTRODUCTION

Recall that a real valued function  $f$  on  $[a, b]$  is said to be Riemann integrable to  $A$  if for every  $\epsilon > 0$  there exists a constant  $\delta > 0$  such that for any division  $D$  of  $[a, b]$  given by

$$a = x_0 < x_1 < \cdots < x_n = b \text{ and } \{\xi_1, \xi_2, \dots, \xi_n\}$$

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with  $x_{i-1} \leq \xi_i \leq x_i$  and  $x_i - x_{i-1} < \delta$  for all  $i$ , we have

$$\left| \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - A \right| < \epsilon. \quad (1)$$

One may see [1], [4], [6], or [8] for further details. The Lebesgue criterion for Riemann integrability states that a function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if and only if  $f$  is bounded and the set of discontinuities of  $f$  has measure 0. In this paper, we define an integral that can integrate bounded and unbounded continuous almost everywhere functions which are Henstock integrable. The idea is to exclude some problematic point-interval pairs of a division in forming the Riemann sum. In turn, this new integral can integrate all improper Riemann integrable functions and even more.

The whole paper is composed of five sections. The second section discusses the basic properties of the  $N$ -integral, including Saks-Henstock Lemma. In the third section, characterizations of the  $N$ -integrable functions will be given by looking at the points of discontinuity. The development of the characterizations utilizes the idea of  $\Gamma_\epsilon$  which was adopted from [2] and [9]. But this time  $\Gamma_\epsilon$  is defined based on discontinuity instead of nondifferentiability. In the fourth section, we explicitly identify the set that would optimally identify the point-interval pairs that are to be excluded in forming the Riemann sums. Finally, in the last section, we present examples to strengthen our results.

## 2. THE $N$ -INTEGRAL

Let  $[a, b]$  be a compact interval in  $\mathbb{R}$ . Given a subset  $X$  of  $[a, b]$ , we denote the closure of  $X$  by  $\bar{X}$ . A partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  is a finite collection of point-interval pairs  $(\xi, [u, v])$  with  $\xi \in [u, v]$ ,  $[u, v] \subset [a, b]$ , and the subintervals  $[u, v]$  are nonoverlapping. If in case the union of the subintervals  $[u, v]$  in  $D$  is  $[a, b]$ , then we simply say that  $D$  is a division of  $[a, b]$ . For a partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$ , we define  $I(D)$  such that

$$I(D) = \{[u, v] : (\xi, [u, v]) \in D\}.$$

Given a subset  $S$  of  $[a, b]$ , a partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  is said to be  $S$ -tagged if for each  $(\xi, [u, v]) \in D$ ,  $\xi \in S$ . A gauge on a set  $X$  is a function from  $X$  to the set of positive real numbers. Given a gauge  $\delta$  on  $[a, b]$ , a point-interval pair  $(\xi, [u, v])$  is said to be  $\delta$ -fine if  $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ . A partial division  $D = \{(\xi, [u, v])\}$  is said to be  $\delta$ -fine if every point-interval pair  $(\xi, [u, v]) \in D$  is  $\delta$ -fine. Note that for two gauges  $\delta_1$  and  $\delta_2$ , with  $\delta_1(x) \leq \delta_2(x)$  for all  $x \in [a, b]$ , every  $\delta_1$ -fine partial division of  $[a, b]$  is also  $\delta_2$ -fine. Recall that a function  $f$  on  $[a, b]$  is said to be Henstock integrable if there exists a real number  $A$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine division  $D = \{(\xi, [u, v])\}$  we have

$$\left| (D) \sum f(\xi)(v - u) - A \right| < \epsilon.$$

See for example [1], [4], [8] or [6]. Here  $A$  is the integral of  $f$  on  $[a, b]$ ; it is unique, and we write

$$(H) \int_a^b f = A.$$

If a function  $f$  is Henstock integrable on  $[a, b]$  then it is Henstock integrable on any subinterval  $[u, v]$  of  $[a, b]$ . It follows that given a Henstock integrable function  $f$  on  $[a, b]$ , we can define a function  $F$  by  $F(a) = 0$  and

$$F(x) = (H) \int_a^x f \text{ for } x \in (a, b).$$

The function  $F$  is called the primitive of  $f$ . A Henstock primitive satisfies the Strong Lusin ( $SL$ ) condition [8]. More precisely, for a Henstock primitive  $F$ , given a set  $S$  of measure zero and  $\epsilon > 0$ , there exists a gauge  $\delta$  on  $[a, b]$  such that for all  $\delta$ -fine  $S$ -tagged partial division  $D$  of  $[a, b]$  we have  $(D) \sum |F(u, v)| < \epsilon$  where  $F(u, v) = F(v) - F(u) = \int_u^v f$ . By a null set, we mean a set of measure zero or empty. Given a null set  $S$  in  $[a, b]$  and a partial division  $D$  of  $[a, b]$ , we set  $\phi_S(D)$  as

$$\phi_S(D) = \{(\xi, [u, v]) \in D : \xi \notin S\}$$

and  $D_S$  as

$$D_S = \{(\xi, [u, v]) \in D : \xi \in S\}.$$

For any partial division  $D$  of  $[a, b]$  define the partial division  $D^* = \{(\xi^*, [u, v])\}$  where  $[u, v]$  comes from  $D$ ,  $\xi^* \in [u, v]$ , and

$$I(D^*) = I(D).$$

In this case, we say that  $D^*$  is  $D$ -compatible or we simply write  $D^* \sim D$ . The term  $D$ -compatible was first used by Lee in her thesis [5]. We shall now state the definition of  $N$ -integral.

**Definition 2.1.** A function  $f$  on  $[a, b]$  is said to be  $N$ -integrable if there exists a null set  $S$  and a real number  $A$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  and any  $D^* = \{(\xi^*, [u, v])\} \sim \phi_S(D)$  we have

$$\left| (D^*) \sum f(\xi^*)(v - u) - A \right| < \epsilon.$$

Here,  $A$  is the  $N$ -integral of  $f$  over  $[a, b]$  and we write

$$\int_a^b f = A.$$

A null set  $S$  satisfying the  $N$ -integrability condition for  $f$ , is said to be an *avoided set* for  $f$  on  $[a, b]$ .

Given a measure zero subset  $S$  of  $[a, b]$  and a division  $D$  of  $[a, b]$ ,  $\phi_S(D)$  may not be a division of  $[a, b]$ . But given  $\alpha > 0$ , there exists a gauge  $\delta$  such that for any

$\delta$ -fine division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$

$$\left| [a, b] \setminus \left( \bigcup_{(\xi, [u, v]) \in \phi_S(D)} [u, v] \right) \right| < \alpha.$$

This follows from the fact that, for a measure zero set  $S$  and  $\alpha > 0$ , there exists an open set  $\mathcal{O}$  containing  $S$  whose measure is less than  $\alpha$ . From this fact also follows that for a function  $f$ , if  $S$  is a set of measure zero and  $\epsilon > 0$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine  $S$ -tagged partial division  $D_S$  of  $[a, b]$  we have  $(D_S) \sum |f(\xi)| |v - u| < \epsilon$ . Given a division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  define  $\|D\| = \max\{|v - u| : (\xi, [u, v]) \in D\}$ . Recall that a function  $f$  on  $[a, b]$  is said to be Riemann integrable if there exists a real number  $A$  for which given  $\epsilon > 0$  there exists a positive number  $\delta_\epsilon$  such that for any division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  with  $\|D\| < \delta_\epsilon$  we have

$$\left| (D) \sum f(\xi)(v - u) - A \right| < \epsilon.$$

See for example [1, 6]. Let  $\epsilon > 0$  and choose such positive number  $\delta_\epsilon$ . Define a constant gauge  $\delta$  on  $[a, b]$  such that for each  $\xi \in [a, b]$ ,  $\delta(\xi) = \delta_\epsilon/2$ . If  $D = \{(\xi, [u, v])\}$  is a  $\delta$ -fine division of  $[a, b]$  and  $D^* \sim D$  then  $\|D\| = \|D^*\| < \delta_\epsilon$ . It follows that

$$\left| (D^*) \sum f(\xi^*)(v - u) - A \right| < \epsilon.$$

We conclude this by the following theorem.

**Theorem 2.2.** *Let  $f$  be a function on  $[a, b]$ . If  $f$  is Riemann integrable then  $f$  is  $N$ -integrable.*

**Theorem 2.3.** *Let  $f$  be an  $N$ -integrable function on  $[a, b]$ . Then  $f$  is Henstock integrable on  $[a, b]$ .*

PROOF. Let  $f$  be  $N$ -integrable on  $[a, b]$ ,  $A$  its integral,  $S$  be an avoided set for  $f$  on  $[a, b]$ , and  $\epsilon > 0$ . Then there exists a gauge  $\delta_1$  such that for any  $\delta_1$ -fine division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have

$$\left| (\phi_S(D)) \sum f(\xi)(v - u) - A \right| < \epsilon.$$

Since  $S$  is of measure zero, it follows from the discussion above that there exists a gauge  $\delta_2$  such that for any  $\delta_2$ -fine  $S$ -tagged partial division  $D_S$  of  $[a, b]$  we have  $(D_S) \sum |f(\xi)| |v - u| < \epsilon$ . Now define  $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ . Then for any  $\delta$ -fine division  $D$  of  $[a, b]$  we have

$$\begin{aligned} \left| (D) \sum f(\xi)(v - u) - A \right| &\leq \left| (\phi_S(D)) \sum f(\xi)(v - u) - A \right| \\ &\quad + (D_S) \sum |f(\xi)| |v - u| \\ &< 2\epsilon. \end{aligned}$$

Therefore  $f$  is Henstock integrable.

The idea of taking the minimum of two gauges, as seen in the proof of Theorem 2.3, has been very useful throughout this paper, even if there are times that the process of doing it is not mentioned anymore.

**Theorem 2.4** (Cauchy-Criterion). *Let  $f$  be a function on  $[a, b]$ . Then  $f$  is  $N$ -integrable if and only if there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine divisions  $D$  and  $P$ , for any  $D^* \sim \phi_S(D)$  and  $P^* \sim \phi_S(P)$ , we have*

$$\left| (D^*) \sum f(\xi^*)(v-u) - (P^*) \sum f(\xi^*)(v-u) \right| < \epsilon.$$

The following theorems can be shown.

**Theorem 2.5.** *Let  $f$  be a function on  $[a, b]$ .*

- (1) *If  $f$  is  $N$ -integrable on  $[a, b]$ , then  $f$  is  $N$ -integrable on every subinterval of  $[a, b]$ .*
- (2) *For  $c \in (a, b)$ , if  $f$  is  $N$ -integrable on each of the intervals  $[a, c]$  and  $[c, b]$ , then  $f$  is  $N$ -integrable on  $[a, b]$  and  $\int_a^c f + \int_c^b f = \int_a^b f$ .*

The following theorem states the linearity of the  $N$ -integral.

**Theorem 2.6.** *Let  $f$  and  $g$  be  $N$ -integrable functions on  $[a, b]$  and  $k \in \mathbb{R}$ . Then*

- (1) *The function  $kf$  is  $N$ -integrable on  $[a, b]$  and  $\int_a^b kf = k \int_a^b f$ , and*
- (2) *The function  $f + g$  is  $N$ -integrable and  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ .*

**Theorem 2.7.** *Let  $f$  be an  $N$ -integrable function on  $[a, b]$  and  $X$  be an avoided set for  $f$ . If  $S$  is a null set containing  $X$  then  $S$  is also an avoided set for  $f$ .*

PROOF. Let  $X$  be an avoided set for an  $N$ -integrable function  $f$  on  $[a, b]$ ,  $S$  be a subset of  $[a, b]$  with measure zero containing  $X$ , and  $\epsilon > 0$ . Then there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine division  $D$  of  $[a, b]$ , and any  $D^* \sim \phi_X(D)$  we have

$$\left| (D^*) \sum f(\xi^*)(v-u) - \int_a^b f \right| < \epsilon.$$

Further since  $S$  is of measure zero, we may assume that for any  $\delta$ -fine  $S$ -tagged partial division  $D_S$  of  $[a, b]$  we have

$$(D_S) \sum |f(\xi)| |v-u| < \epsilon.$$

Now let  $D$  be any  $\delta$ -fine division of  $[a, b]$ ,  $D^* \sim \phi_S(D)$  and  $D_{S \setminus X} = \{(\xi, [u, v]) \in D : \xi \in S \setminus X\}$ . Then  $(D^* \cup D_{S \setminus X}) \sim \phi_X(D)$  and

$$\begin{aligned} \left| (D^*) \sum f(\xi^*)(v-u) - \int_a^b f \right| &\leq \left| (D^* \cup D_{S \setminus X}) \sum f(\xi^*)(v-u) - \int_a^b f \right| \\ &\quad + (D_{S \setminus X}) \sum |f(\xi)| |v-u| \\ &< 2\epsilon. \end{aligned}$$



**Theorem 2.8** (Saks-Henstock Lemma). *Let  $f$  be an  $N$ -integrable function on  $[a, b]$  with primitive  $F$ . Then there exists a null set  $S$  such that given  $\epsilon > 0$ , there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine division  $D$  of  $[a, b]$  and any  $D^* \sim \phi_S(D)$  we have*

$$(D^*) \sum |f(\xi^*)(v - u) - F(u, v)| < \epsilon.$$

PROOF. Let  $f$  be  $N$ -integrable. Then  $f$  is Henstock integrable and the primitive  $F$  of  $f$  satisfies the Strong Lusin Condition (SL). Let  $S$  be an avoided set for  $f$  on  $[a, b]$  and  $\epsilon > 0$ . Then there exists gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine division  $D$  of  $[a, b]$  and any  $D^* \sim \phi_S(D)$  we have

$$\left| (D^*) \sum f(\xi^*)(v - u) - F(a, b) \right| < \frac{\epsilon}{8}.$$

Let  $\epsilon > 0$  and choose such  $\delta$ . Since  $F$  is SL we may assume that for any  $\delta$ -fine  $S$ -tagged partial division  $D$  we have

$$(D) \sum |F(u, v)| < \frac{\epsilon}{4}.$$

Now, let  $D$  be a  $\delta$ -fine division of  $[a, b]$  and  $D^* \sim \phi_S(D)$ . Split  $D^*$  into  $D_1$  and  $D_2$  such that  $D_1$  contains those pairs  $(\xi^*, [u, v])$  with  $f(\xi^*)(v - u) - F(u, v) < 0$ . Let  $D_S = \{(\xi, [u, v]) \in D : \xi \in S\}$ . We will show that  $(D_i) \sum |f(\xi^*)(v - u) - F(u, v)| < \frac{\epsilon}{4}$  for  $i = 1, 2$ . For  $D_2$ , let  $D_2^0 = \bigcup_{(\xi^*, [u, v]) \in D_2} [u, v]$ . For  $D_S$ , let  $D_S^0 = \bigcup_{(\xi, [u, v]) \in D_S} [u, v]$ . The subset  $\overline{[a, b] \setminus (D_2^0 \cup D_S^0)}$  of  $[a, b]$  is a finite union of subintervals of  $[a, b]$ . We know that for each interval in  $\overline{[a, b] \setminus (D_2^0 \cup D_S^0)}$ ,  $f$  is  $N$ -integrable. It follows that there exists a gauge  $\delta_2 \leq \delta$  such that for any  $\delta_2$ -fine division  $B$  of  $\overline{[a, b] \setminus (D_2^0 \cup D_S^0)}$  and any  $B^* \sim \phi_S(B)$  we have

$$\left| (B^*) \sum f(\xi^*)(v - u) - (B) \sum F(u, v) \right| < \frac{\epsilon}{8}.$$

Let  $B$  be a  $\delta_2$ -fine division of  $\overline{[a, b] \setminus (D_2^0 \cup D_S^0)}$ ,  $B^* \sim \phi_S(B)$ , and  $C$  be the subset of  $\phi_S(D)$  such that  $D_2 \sim C$ . Based on how  $B$  is obtained,  $B$  and  $D_2$  are nonoverlapping. Furthermore, notice that the union  $C \cup B \cup D_S$  forms a  $\delta$ -fine division of  $[a, b]$ . Denote the union  $C \cup B \cup D_S$  by  $Q$ . Since  $B^* \sim \phi_S(B)$  and  $D_2 \sim C$ , it follows that  $(D_2 \cup B^*)$  is  $\phi_S(Q)$ -compatible. Therefore

$$\begin{aligned} (D_2) \sum |f(\xi^*)(v - u) - F(u, v)| &= \left| (D_2) \sum \{f(\xi^*)(v - u) - F(u, v)\} \right| \\ &= \left| (D_2) \sum f(\xi^*)(v - u) - (C) \sum F(u, v) \right| \\ &\leq \left| (D_2 \cup B^*) \sum f(\xi^*)(v - u) - F(a, b) \right| \\ &\quad + \left| (B^*) \sum f(\xi^*)(v - u) - (B) \sum F(u, v) \right| \\ &\quad + (D_S) \sum |F(u, v)| \\ &< \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

A similar argument can be made for  $D_1$ .

### 3. THE $N$ -INTEGRABLE FUNCTIONS

In this section, we identify which among the Henstock integrable functions are  $N$ -integrable. For a partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  define the double-tagged partial division  $D^{**} = \{(\{s, t\}, [u, v])\}$  where  $[u, v]$  comes from  $D$ ,  $s, t \in [u, v]$ , and the union of intervals in  $D$  and  $D^{**}$  are the same. In this case we say that  $D^{**}$  is  $D$ -double-tagged compatible or we simply write  $D^{**} \approx D$ . Given a function  $f$  on  $[a, b]$  and  $\epsilon > 0$  we set

$$\Gamma_\epsilon^{**} = \{(\{s, t\}, [u, v]) : |f(s) - f(t)| \geq \epsilon, s, t \in [u, v] \text{ and } [u, v] \subset [a, b]\}.$$

Note that the  $\Gamma_\epsilon^{**}$  is dependent only on  $f$  and  $\epsilon$ .

**Theorem 3.1.** *Let  $f$  be a Henstock integrable function on  $[a, b]$  and  $F$  be its primitive. Then  $f$  is  $N$ -integrable if and only if there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$ , and any  $\phi_S(D)$ -double-tagged compatible partial division  $D^{**} = \{(\{s, t\}, [u, v])\}$  we have*

$$(D^{**} \cap \Gamma_\epsilon^{**}) \sum |f(s) - f(t)||v - u| < \epsilon.$$

PROOF. Let  $f$  be an  $N$ -integrable function on  $[a, b]$ ,  $S$  be an avoided set for  $f$  on  $[a, b]$  and  $\epsilon > 0$ . Then there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D$  of  $[a, b]$  and any  $D^* \sim \phi_S(D)$  we have

$$(D^*) \sum |f(\xi^*)(v - u) - F(u, v)| < \epsilon.$$

Choose such a gauge  $\delta$  and let  $D = \{(\xi, [u, v])\}$  be a  $\delta$ -fine partial division of  $[a, b]$ . If  $D^{**} = \{(\{s, t\}, [u, v])\}$  is a  $\phi_S(D)$ -double-tagged compatible partial division of  $[a, b]$ , then  $D_1^* = \{(s, [u, v]) : (\{s, t\}, [u, v]) \in D^*\}$  and  $D_2^* = \{(t, [u, v]) : (\{s, t\}, [u, v]) \in D^*\}$  are both  $\phi_S(D)$ -compatible. It follows that

$$\begin{aligned} (D^{**} \cap \Gamma_\epsilon^{**}) \sum |f(s) - f(t)||v - u| &\leq (D^{**} \cap \Gamma_\epsilon^{**}) \sum |f(s)(v - u) - F(u, v)| \\ &\quad + (D^{**} \cap \Gamma_\epsilon^{**}) \sum |F(u, v) - f(t)(v - u)| \\ &< \epsilon + \epsilon \\ &= 2\epsilon. \end{aligned}$$

For the converse, there exists a null set  $S$  such that given  $\epsilon > 0$ , there exists a gauge  $\delta$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  and any  $\phi_S(D)$ -double-tagged compatible partial division  $D^{**} = \{(\{s, t\}, [u, v])\}$  we have

$$(D^{**} \cap \Gamma_\epsilon^{**}) \sum |f(s) - f(t)||v - u| < \epsilon.$$

Let  $\epsilon > 0$  be given and choose such a gauge  $\delta$ . Since  $f$  is Henstock integrable, we may assume that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have

$$(D) \sum \left| f(\xi)(v - u) - (H) \int_u^v f \right| < \epsilon \text{ and } (D_S) \sum |f(\xi)||v - u| < \epsilon.$$

Now, let  $D = \{(\xi, [u, v])\}$  be a  $\delta$ -fine division of  $[a, b]$  and  $D^* = (\xi^*, [u, v]) \sim \phi_S(D)$ . Note that the double-tagged partial division  $D^{**} = \{(\xi, \xi^*), [u, v]\}$  is  $\phi_S(D)$ -double-tagged compatible and

$$\begin{aligned} |(D^*) \sum f(\xi^*)(v - u) - (\phi_S(D)) \sum f(\xi)(v - u)| \\ \leq (D^{**}) \sum |f(\xi^*) - f(\xi)| |v - u| \\ \leq (D^{**} \setminus \Gamma_\epsilon^{**}) \sum |f(\xi^*) - f(\xi)| |v - u| \\ + (D^{**} \cap \Gamma_\epsilon^{**}) \sum |f(\xi^*) - f(\xi)| |v - u|. \end{aligned}$$

Then

$$\begin{aligned} \left| (D^*) \sum f(\xi^*)(v - u) - (H) \int_a^b f \right| \\ \leq \left| (D^*) \sum f(\xi^*)(v - u) - (\phi_S(D)) \sum f(\xi)(v - u) \right| \\ + \left| (\phi_S(D) \cup D_S) \sum f(\xi)(v - u) - (H) \int_a^b f \right| \\ + (D_S) \sum |f(\xi)||v - u| \\ < \epsilon(b - a) + \epsilon + \epsilon + \epsilon \\ = \epsilon(b - a + 2). \end{aligned}$$

In what follows, given a function  $f$  on  $[a, b]$  and a positive number  $\epsilon$ ,

$$\Gamma_\epsilon = \{(\xi, [u, v]) : |f(v) - f(u)| \geq \epsilon, \xi \in [u, v] \text{ and } [u, v] \subset [a, b]\}.$$

Similar to  $\Gamma_\epsilon^{**}$ ,  $\Gamma_\epsilon$  depends only on  $f$  and  $\epsilon$ .

**Theorem 3.2.** *Let  $f$  be an  $N$ -integrable function on  $[a, b]$ . Then there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D$  of  $[a, b]$*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v - u| < \epsilon.$$

PROOF. This follows immediately from Theorem 3.1.

**Theorem 3.3.** *Let  $f$  be a function on  $[a, b]$ . If there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v - u| < \epsilon$$

*then  $f$  is continuous almost everywhere on  $[a, b]$ .*

PROOF. Let  $D_f$  be the set of discontinuity of  $f$  in  $[a, b] \setminus S$ . Assume that  $a, b \in S$ . If  $x \in D_f$ , there exists  $\eta(x) > 0$  such that for all  $s > 0$  there exists a closed interval  $J_{x,s}$  whose one endpoint is  $x$  with  $J_{x,s} \subset (x - s, x + s)$  and that  $|f(J_{x,s})| > \eta(x)$ . Here  $f(I) = |f(v) - f(u)|$  whenever  $I = [u, v]$ . Fix a positive integer  $n$  and let  $E_n = \{x \in [a, b] : \eta(x) \geq \frac{1}{n}\}$ . We will show that for each  $n$ ,  $E_n$  is of Lebesgue measure zero. Given  $\epsilon > 0$ , there exists a gauge  $\delta$  mapping  $[a, b]$  to  $(0, 1)$  such that for any  $\delta$ -fine partial division  $D$  of  $[a, b]$  we have

$$(D \cap \Gamma_{\frac{\epsilon}{n}}) \sum |f(I_x)||I_x| < \frac{\epsilon}{n}$$

Let  $\mathcal{S}_n = \{J_{x,s} : x \in E_n, 0 < s < \delta(x)\}$ . Note that  $\mathcal{S}_n$  is a Vitali cover for  $E_n$ . By the Vitali covering theorem there are pairwise disjoint intervals  $J_1, J_2, \dots, J_k$  in  $\mathcal{S}_n$  and sequence of closed intervals  $\{J_i\}_{i=k+1}^\infty \subset \mathcal{S}_n$  such that

$$E_n \subset \bigcup_{i=1}^k J_i \cup \bigcup_{i=k+1}^\infty J_i \text{ and } \sum_{i=k+1}^\infty |J_i| \leq \epsilon.$$

Note that  $D = \{(x, J_i)\}_{i=1}^k$  is a  $\delta$ -fine partial division of  $[a, b]$  and for each  $J_i \in D$ ,  $J_i = J_{x,s}$  for some  $J_{x,s}$  in  $\mathcal{S}_n$ . Hence

$$\begin{aligned} \frac{1}{n}(D) \sum |J_i| &= \frac{1}{n}(D) \sum |J_{x,s}| \\ &\leq (D) \sum \eta(x)|J_{x,s}| \\ &< (D \setminus \Gamma_{\frac{\epsilon}{n}}) \sum |f(J_{x,s})||J_{x,s}| + (D \cap \Gamma_{\frac{\epsilon}{n}}) \sum |f(J_{x,s})||J_{x,s}| \\ &\leq \frac{\epsilon}{n}(b - a) + \frac{\epsilon}{n} \end{aligned}$$

It follows that  $\mu(E_n) < \epsilon(2 + b - a)$ . Since  $\epsilon$  is arbitrary we conclude that  $E_n$  is of measure zero. Since  $D_f$  is a countable union of measure zero sets  $E_n$ ,  $D_f$  is of measure zero. Since  $D_f$  and  $S$  are of measure zero, we conclude that  $f$  is continuous almost everywhere on  $[a, b]$ .

**Theorem 3.4.** *Let  $f$  be a Henstock integrable function on  $[a, b]$ . If  $f$  is continuous almost everywhere on  $[a, b]$  then  $f$  is  $N$ -integrable.*

PROOF. Let  $f$  be continuous almost everywhere on  $[a, b]$  and Henstock integrable. Let  $D_f$  be the set of discontinuity of  $f$  on  $[a, b]$ . Let  $\epsilon > 0$ . Since  $f$  is Henstock integrable, there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have

$$(D) \sum \left| f(\xi)(v - u) - (H) \int_u^v f \right| < \epsilon \text{ and } (D_{D_f}) \sum \left| (H) \int_u^v f \right| < \epsilon$$

where  $D_{D_f} = \{(\xi, [u, v]) \in D : \xi \in D_f\}$ . By continuity of  $f$  on each element of  $[a, b] \setminus D_f$ , we may assume that for every  $\delta$ -fine pair  $(\xi, [u, v])$  with  $\xi \in [a, b] \setminus D_f$

$$|f(\xi) - f(\xi^*)| < \epsilon$$

whenever  $\xi^* \in [u, v]$ . So for any  $\delta$ -fine division  $D$  of  $[a, b]$  and  $D^* \sim \phi_{D_f}(D)$  we have

$$\begin{aligned}
& \left| (D^*) \sum f(\xi^*)(v-u) - (H) \int_a^b f \right| \\
& \leq \left| (D^*) \sum f(\xi^*)(v-u) - (\phi_{D_f}(D)) \sum (H) \int_u^v f \right| \\
& \quad + (D_{D_f}) \sum \left| (H) \int_u^v f \right| \\
& \leq (\phi_{D_f}(D)) \sum |f(\xi^*) - f(\xi)| |v-u| \\
& \quad + (\phi_{D_f}(D)) \sum \left| f(\xi)(v-u) - (H) \int_u^v f \right| \\
& \quad + (D_{D_f}) \sum \left| (H) \int_u^v f \right| \\
& < \epsilon(b-a) + \epsilon + \epsilon.
\end{aligned}$$

Combining Theorem 3.2, Theorem 3.3 and Theorem 3.4, we have the following result.

**Theorem 3.5.** *Let  $f$  be a function  $[a, b]$ . Then the following statements are equivalent:*

- (1) *The function  $f$  is  $N$ -integrable*
- (2) *The function  $f$  is Henstock integrable on  $[a, b]$  and there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)| |v-u| < \epsilon$$

and

- (3) *The function  $f$  is Henstock integrable on  $[a, b]$  and continuous almost everywhere.*

#### 4. THE AVOIDED SET

For a Riemann integrable function  $f$ , the avoided set may be empty, even if  $f$  has discontinuities. So it is natural to ask what is the optimal avoided set for an  $N$ -integrable function. We answer this query in this section. The ideas here will also able us to present  $N$ -integral as a generalization of improper Riemann integral. In what follows, for a function  $f$  on  $[a, b]$ , we set

$$S_\infty = \{x \in [a, b] : \text{for some } \{x_n\} \text{ in } [a, b], x_n \rightarrow x \text{ and } f(x_n) \rightarrow \pm\infty\}.$$

Note that for any function  $f$  on  $[a, b]$ ,  $S_\infty$  is closed.

**Lemma 4.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be  $N$ -integrable and  $S$  be an avoided set for  $f$ . Then  $S_\infty \subset S$ .*

PROOF. Let  $x$  be in  $S_\infty$  but not in  $S$  and  $\delta$  be a gauge on  $[a, b]$ . Choose a  $\delta$ -fine pair  $(x, [u, v])$  such that  $f$  is not bounded on  $[u, v]$ . Note that since  $(x, [u, v])$  is  $\delta$ -fine, it is an element of some  $\delta$ -fine division  $D$  of  $[a, b]$  with  $(x, [u, v]) \in \phi_S(D)$ .

The result below can be easily shown.

**Lemma 4.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  and  $X$  be a subset of  $[a, b]$  be of measure with  $X \cap S_\infty = \emptyset$ . Then given  $\epsilon > 0$ , there exists a gauge  $\delta$  such that for any  $\delta$ -fine  $X$ -tagged partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  and any  $D^* = \{(\xi^*, [u, v])\}$  with  $I(D^*) = I(D)$  we have*

$$(D^*) \sum |f(\xi^*)|(v - u) < \epsilon.$$

**Theorem 4.3.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  and suppose that  $S_\infty$  is of measure zero. Then  $f$  is  $N$ -integrable if and only if there exists a real number  $A$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$ , and any  $D^* = \{(\xi^*, [u, v])\} \sim \phi_{S_\infty}(D)$ , we have*

$$\left| (D^*) \sum f(\xi^*)(v - u) - A \right| < \epsilon.$$

PROOF. Suppose  $f$  is  $N$ -integrable. Then there exist a subset  $S$  of  $[a, b]$  of measure zero and a real number  $A$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$ , and any  $D^* = \{(\xi^*, [u, v])\} \sim \phi_S(D)$  we have

$$\left| (\phi_S(D)) \sum f(\xi^*)(v - u) - A \right| < \epsilon.$$

From the previous lemma, we may assume that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  whose tags are in  $S \setminus S_\infty$  and any  $D^* = \{(\xi^*, [u, v])\} \sim D$ , we have

$$(D^*) \sum |f(\xi^*)|(v - u) < \epsilon.$$

Now, let  $D = \{(\xi, [u, v])\}$  be a  $\delta$ -fine division of  $[a, b]$  and  $D^* = \{(\xi^*, [u, v])\} \sim \phi_{S_\infty}(D)$ . Set

$$D_{S \setminus S_\infty} = \{(\xi, [u, v]) \in D : \xi \in S \text{ and } \xi \notin S_\infty\}.$$

Note that  $I(\phi_S(D)) = I(D^* \setminus D_{S \setminus S_\infty}^*)$ , where  $D_{S \setminus S_\infty}^*$  is the subset of  $D^*$  such that

$$I(D_{S \setminus S_\infty}^*) = I(D_{S \setminus S_\infty}).$$

It follows that

$$\begin{aligned} \left| (D^*) \sum f(\xi^*)(v - u) - A \right| &\leq \left| (D^* \setminus D_{S \setminus S_\infty}^*) \sum f(\xi^*)(v - u) - A \right| \\ &\quad + (D_{S \setminus S_\infty}^*) \sum |f(\xi^*)|(v - u) \\ &< 2\epsilon. \end{aligned}$$

The converse follows from the definition.

The importance of stating the result below is that it presents the  $N$ -integral as a generalization of improper Riemann integral.

**Theorem 4.4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$ . Then  $f$  is  $N$ -integrable if and only if there exists a real number  $A$  and a subset  $S$  of  $[a, b]$  of measure zero for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$ ,*

and any  $D^* = \{(\xi^*, [u, v])\} \sim \phi_S(D)$  we have  $[u, v] \cap S = \emptyset$  for each  $(\xi, [u, v]) \in \phi_S(D)$ ,

$$(b - a) - (\phi_S(D)) \sum (v - u) < \epsilon$$

and

$$\left| (D^*) \sum f(\xi^*)(v - u) - A \right| < \epsilon.$$

In this case, we may choose  $S = S_\infty$ .

### 5. SOME EXAMPLES

Recall that the function  $f$  on  $[0, 1]$  defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

is nowhere continuous. This function is called the Dirichlet function. It follows from Theorem 3.5 that this function, while Henstock integrable, is not  $N$ -integrable. Moreover the function

$$f(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

while not Riemann integrable, is  $N$ -integrable. Therefore the converses of Theorem 2.3 and Theorem 2.2 do not hold. Before we go to our examples, we shall present a characterization of continuous almost everywhere functions.

**Theorem 5.1.** *A function  $f$  on  $[a, b]$  is continuous almost everywhere if and only if there exists a null set  $S$  for which given  $\epsilon > 0$  there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine partial division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$  we have*

$$(\phi_S(D) \cap \Gamma_\epsilon) \sum |f(v) - f(u)||v - u| < \epsilon.$$

**PROOF.** Let  $f$  be continuous almost everywhere on  $[a, b]$ . Then there exists a null set  $S$  in  $[a, b]$  such that  $f$  is continuous on every  $x \in [a, b] \setminus S$ . Let  $\epsilon > 0$  be given. It follows that there exists a gauge  $\delta$  on  $[a, b]$  such that for any  $\delta$ -fine pair  $(\xi, [u, v])$  with  $\xi \in [a, b] \setminus S$ , we have  $|f(v) - f(u)| < \epsilon$ . Choose such gauge  $\delta$ . Then for any  $\delta$ -fine division  $D = \{(\xi, [u, v])\}$  of  $[a, b]$ , we have  $\phi_S(D) \cap \Gamma_\epsilon = \emptyset$ . The converse follows from Theorem 3.3.

A partial partition  $D = \{[u, v]\}$  is a finite collection of nonoverlapping subintervals of  $[a, b]$ . Given a subset  $X$  of  $[a, b]$ , a partial partition  $D$  is said to be  $X$ -vertexed if for each  $[u, v] \in D$ , either  $u \in X$  or  $v \in X$ . In [6], a function  $f$  on  $[a, b]$  is said to be  $BV^*(X)$  if there exists a nonnegative number  $M$  such that for any  $X$ -vertexed partial partition  $D$  of  $[a, b]$  we have  $(D) \sum |f(v) - f(u)| \leq M$ . The function  $f$  is  $BVG^*$  on  $[a, b]$  if  $[a, b] = \bigcup_{i=1}^\infty X_i$  such that for each  $i$ , there is a positive number  $M_i$  such that for any  $X_i$ -vertexed partial partition  $D$  of  $[a, b]$  we have  $(D) \sum |f(v) - f(u)| \leq M_i$ . In this case we say that  $f$  is  $BVG^*$  on  $[a, b]$  using  $\{(X_i, M_i)\}$ .

**Theorem 5.2.** *If a function  $f$  on  $[a, b]$  is  $BVG^*$  on  $[a, b]$  then it is continuous almost everywhere.*

PROOF. Let  $\epsilon > 0$  and suppose that  $f$  is  $BVG^*$  using  $\{(X_i, M_i)\}$ . For each  $i$ , put  $\delta(x) \leq \eta_i$  for all  $x \in X_i$  where  $\eta_i = \frac{\epsilon}{M_i 2^{i+1}}$ . Let  $D$  be  $\delta$ -fine a partial division of  $[a, b]$ . For each  $i$ , let

$$D_i = \{(\xi, [u, v]) \in D : \xi \in X_i\}.$$

Then

$$\begin{aligned} (D \cap \Gamma_\epsilon) \sum |f(v) - f(u)|(v - u) &= \sum_{i=1}^{\infty} (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(u)|(v - u) \\ &< \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(u)| \\ &\leq \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(v) - f(\xi)| \\ &\quad + \sum_{i=1}^{\infty} \eta_i (D_i \cap \Gamma_\epsilon) \sum |f(\xi) - f(u)| \\ &\leq \sum_{i=1}^{\infty} \eta_i M_i + \sum_{i=1}^{\infty} \eta_i M_i \\ &< \epsilon. \end{aligned}$$

The next result follows from Theorem 3.5 and Theorem 5.2.

**Theorem 5.3.** *Let  $f$  be a Henstock integrable function on  $[a, b]$ . If  $f$  is  $BVG^*$  on  $[a, b]$  then  $f$  is  $N$ -integrable.*

A function  $f$  on  $[a, b]$  is Baire class one or of first Baire class or simply Baire one if it is a pointwise limit of a sequence of continuous functions on  $[a, b]$ . Recently, in [3], a Baire one function  $f$  was characterized by topologically describing its set of discontinuity ( $D_f$ ) and how it behaves on this set. As a consequence, it follows from Theorem [3, Theorem 3.4] that if for a function  $f$ ,  $f(x) = 0$  for  $x \in D_f$ , then  $f$  is Baire one. Therefore we have the following result showing the closeness of  $N$ -integrable functions to Baire one functions.

**Theorem 5.4.** *Let  $f$  be a function on  $[a, b]$  and suppose that  $D_f$  is of measure zero. If  $\overline{D_f}$  is of measure zero then there exists a Baire one function  $g$  on  $[a, b]$  such that  $g = f$  almost everywhere on  $[a, b]$ .*

Finally, we present an example whose set of discontinuity is of measure zero but not countable. Consider the function  $F$  defined by

$$F(x) = \begin{cases} (x - a)^2 \sin\left(\frac{1}{x-a}\right)^2 & \text{if } x \in (a, b), \\ 0 & \text{if } x = a. \end{cases}$$



The function  $F$  is differentiable at every point in  $[a, b]$ ,  $F'(a) = 0$  and

$$F'(x) = 2(x - a) \sin\left(\frac{1}{(x - a)^2}\right) - \frac{2}{x - a} \cos\left(\frac{1}{(x - a)^2}\right)$$

for  $x \in (a, b]$ . Now, consider the Cantor set  $\mathcal{C}$ . Recall that the Cantor set is nowhere dense, uncountable, and of measure zero. Let  $[a, b] \setminus \mathcal{C} = \cup_{n=1}^{\infty} (a_n, b_n)$ . For each  $n$ , consider

$$F_{1,n}(x) = \begin{cases} (x - a_n)^2 \sin\left(\frac{1}{x - a_n}\right)^2 & \text{if } x \in (a_n, b_n], \\ 0 & \text{if } x = a_n \end{cases}$$

and

$$F_{2,n}(x) = \begin{cases} (b_n - x)^2 \sin\left(\frac{1}{b_n - x}\right)^2 & \text{if } x \in [a_n, b_n), \\ 0 & \text{if } x = b_n. \end{cases}$$

Note that both  $F_{1,n}$  and  $F_{2,n}$  are differentiable on  $[a_n, b_n]$  and that the graph of  $F_{1,n}$  is just the reflection of  $F_{2,n}$  with respect to  $(\frac{a_n + b_n}{2}, \frac{a_n + b_n}{2})$ . From  $(a_n, \frac{a_n + b_n}{2})$ , choose  $c_n$  such that  $F'_{1,n}(c_n) = 0$ . Then choose  $d_n$  such that  $c_n - a_n = b_n - d_n$ . Hence

$$F_{1,n}(c_n) = (c_n - a_n)^2 \sin\left(\frac{1}{c_n - a_n}\right)^2 = (b_n - d_n)^2 \sin\left(\frac{1}{d_n - a_n}\right)^2 = F_{2,n}(d_n).$$

Note also that, by symmetry,  $F'_{2,n}(d_n) = 0$ . Now define

$$F_n(x) = \begin{cases} F_{1,n}(x) & \text{if } x \in (a_n, c_n] \\ F_{1,n}(c_n) & \text{if } x \in (c_n, d_n) \\ F_{2,n}(x) & \text{if } x \in [d_n, b_n). \end{cases}$$

If we set  $F_n(a_n) = F_n(b_n) = 0$  then  $F_n$  is differentiable at all points in  $[a_n, b_n]$ . At this point, it is important to note that the the graph of  $F_n$  is symmetric with respect to the midpoint of  $a_n$  and  $b_n$ . Define a function  $G$  on  $[a, b]$  by

$$G(x) = \begin{cases} F_n(x) & \text{if } x \in (a_n, b_n), \\ 0 & \text{if } x \in \mathcal{C}. \end{cases}$$

As constructed, for each  $n$ ,  $G$  is differentiable on  $(a_n, b_n)$ . Furthermore, it can be shown that  $G'(\xi) = 0$  for  $\xi \in \mathcal{C}$ .

At this point we already know that  $G$  is differentiable at any point in  $[a, b]$ . So its derivative  $G'$  is Henstock integrable. Note also that  $G'$  is continuous on  $(a_n, b_n)$  for any  $n$ . We will show that  $G$  is not continuous at any  $\xi \in \mathcal{C}$ . For  $\xi \in \mathcal{C}$ , choose  $\{(a_{n_k}, b_{n_k})\}$  such that

$$a_{n_k} \rightarrow \xi \text{ as } k \rightarrow \infty.$$

For each  $k$ , choose  $m_k > k$  such that

$$a_{n_k} + \frac{1}{\sqrt{2m_k\pi}} \in (a_{n_k}, c_{n_k}]$$

and set

$$x_k = a_{n_k} + \frac{1}{\sqrt{2m_k\pi}}.$$

So  $x_k \rightarrow \xi$  but

$$G'(x_k) = -2\sqrt{2m_k\pi}.$$

We will now show that  $G'$  is  $N$ -integrable. Let  $\epsilon > 0$ . Since  $G'$  is the derivative of  $G$  there exists a gauge  $\delta$  such that for any  $\delta$ -fine pair  $(\xi, [u, v])$  we have

$$|G'(\xi)(v - u) - G(u, v)| < \epsilon(v - u).$$

We may assume that for any  $\delta$ -fine pair  $(\xi, [u, v])$  with  $\xi \in [a, b] \setminus \mathcal{C}$ , we have

$$|G'(\xi) - G'(\xi^*)| < \epsilon \text{ whenever } \xi^* \in [u, v].$$

Now let  $D = \{(\xi, [u, v])\}$  be a  $\delta$ -fine division of  $[a, b]$  and  $D^* \sim \phi_{\mathcal{C}}(D)$ . Split  $D$  into  $D_1$  and  $D_2$  such that  $D_1 = \{(\xi, [u, v]) \in D : \xi \in \mathcal{C}\}$  and  $D_2 = D \setminus D_1$ . It follows that

$$\begin{aligned} \left| (D^*) \sum G'(\xi^*)(v - u) - G(a, b) \right| &\leq (D^*) \sum |G'(\xi^*)(v - u) - G(u, v)| \\ &\quad + (D_1) \sum |G(u, v)| \\ &\leq (\phi_{\mathcal{C}}(D)) \sum |G'(\xi^*) - G'(\xi)|(v - u) \\ &\quad + (\phi_{\mathcal{C}}(D)) \sum |G'(\xi)(v - u) - G(u, v)| \\ &\quad + (D_1) \sum |G(u, v) - G'(\xi)(v - u)| \\ &\quad + (D_1) \sum |G'(\xi)||v - u| \\ &\leq \epsilon(b - a) + \epsilon(b - a) \\ &< 2\epsilon(b - a). \end{aligned}$$

Therefore  $G'$  is  $N$ -integrable on  $[a, b]$ . Furthermore, it can be shown that for the function  $G$ ,  $S_{\infty} = \mathcal{C}$ . The function  $G'$  is neither Lebesgue nor improper Riemann integrable.

Excluding the point-interval pairs which are tagged in the avoided set  $S$  in forming the Riemann sum enables us to integrate not just all Riemann integrable functions but even all improper Riemann integrable functions which are not Lebesgue integrable. In fact, there are  $N$ -integrable functions that are neither Lebesgue nor improper Riemann integrable. Since the discontinuity of an  $N$ -integrable function is  $F_{\sigma}$  and of measure zero, it follows from [10, p. 273] that it is of first category. A natural question that arises from this paper is whether it is possible to develop an integral that integrates functions whose sets of discontinuity are of the first category.

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