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Abstract: A vertex coloring $c : V(G) \to \mathbb{N}$ of a non-trivial graph G is called a *sigma coloring* if $\sigma(u) \neq \sigma(v)$ for any pair of adjacent vertices u and v. Here, $\sigma(x)$ denotes the sum of the colors assigned to vertices adjacent to x. The *sigma chromatic number* of G, denoted by $\sigma(G)$, is defined as the fewest number of colors needed to construct a sigma coloring of G. In this paper, we consider the sigma chromatic number of graphs obtained by deleting one or more of its edges. In particular, we study the difference $\sigma(G) - \sigma(G - e)$ in general as well as in restricted scenarios; here, $G - e$ is the graph obtained by deleting an edge e from G. Furthermore, we study the sigma chromatic number of graphs obtained via multiple edge deletions in complete graphs by considering the complements of paths and cycles.

Keywords: sigma coloring, edge deletion, neighbor-distinguishing coloring, complement

1. Introduction

A neighbor-distinguishing graph coloring is a coloring of the vertices and/or edges of a graph that induces a vertex labelling under which any pair of adjacent vertices is assigned different labels. The most studied example of a neighbor-distinguishing coloring is the well-studied proper vertex coloring. Several neighbor-distinguishing colorings have been introduced and studied in the literature such as in Refs. [2] and [5]. In Ref. [4], Chartrand, Okamoto, and Zhang introduced a new kind of neighbor-distinguishing vertex coloring defined as follows.

Definition 1 (Chartrand et al. [4]). *For a non-trivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G. For each $v \in V(G)$, the* ***color sum*** *of v, denoted by $\sigma(v)$, is defined to be the sum of the colors of the vertices adjacent to v. If $\sigma(u) \neq \sigma(v)$ for every two adjacent $u, v \in V(G)$, then c is called a* ***sigma coloring*** *of G. The minimum number of colors required in a sigma coloring of G is called its* ***sigma chromatic number*** *and is denoted by $\sigma(G)$.*

The notion of sigma coloring is related to the vertex colorings/labellings discussed in Refs. [1], [8], [11]. These colorings/labellings also use the sum of the colors/labels of a vertex's neighbors. Sigma colorings of different families of graphs have already been studied in Refs. [4], [6], and [9].

In this paper, we study the sigma chromatic number in relation to edge deletion. Let $G = (V, E)$ be a graph. Let $\mathcal{V} \subseteq V$ and $\mathcal{E} \subseteq E$. We denote by $G - \mathcal{V}$ the graph obtained by deleting from G all vertices in $\mathcal{V}$ and all edges with at least one end vertex in $\mathcal{V}$. Moreover, we denote by $G - \mathcal{E}$ the graph obtained by deleting from G all edges in $\mathcal{E}$. For simplicity, when $\mathcal{V}$ or $\mathcal{E}$ is a singleton, say $\{k\}$, we denote $G - \mathcal{V}$ or $G - \mathcal{E}$ simply by $G - k$.

Previous work has been done on chromatic numbers in relation to edge deletion. For instance, it is well-known that $0 \leq \chi(G) - \chi(G - e) \leq 1$. In Ref. [10], the notion of critical edges (and vertices) was considered and defined as follows: An edge (or vertex) in a graph is *critical* if its deletion reduces the chromatic number of the graph by one. The paper studied the complexity of the problem of testing for the existence of critical vertices and edges in H-free graphs and showed that an edge in a graph is critical if and only if its contraction reduces the chromatic number by one.

In Ref. [7], b-colorings were studied in relation to edge-deleted subgraphs. A *b-coloring* of a graph G with k colors is a proper coloring of G that uses k colors such that for each color class, there is a vertex that has a neighbor in each of the other color classes. The *b-chromatic number* of G, denoted by $b(G)$, is the largest positive integer k for which G has a b-coloring using k colors. In Ref. [7], it was shown that $b(G) - b(G - e) \geq 2 - \lceil \frac{n}{2} \rceil$.

In Ref. [2], Chartrand et al. studied edge deletion in relation to another neighbor-distinguishing coloring called set coloring. Let $c : V(G) \to \mathbb{N}$ be a vertex coloring of a non-trivial connected graph G and denote by $\mathrm{NC}(x)$ the set of colors assigned to vertices adjacent to x. Then c is called a *set coloring* if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for any pair of adjacent vertices u and v. The *set chromatic number* of G, denoted by $\chi_S(G)$, is defined as the least number of colors needed to construct a set coloring of G. Since a set coloring induces a proper vertex coloring using the neighborhood of each vertex, it is interesting to study the effect of edge deletion (i.e., the removal of a neighbor from two vertices) on the set chromatic number. In Ref. [2], Chartrand et al. proved the following:

Theorem 2 (Ref. [2]).

(1) If e is an edge of a graph G, then

$$|\chi_S(G) - \chi_S(G - e)| \leq 2.$$

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(2) *If $e = uv$ is an edge of a graph G that is not a bridge such that $d_{G-e}(u, v) \geq 4$, then*

$$|\chi_S(G) - \chi_S(G - e)| \leq 1.$$

Since a sigma coloring also induces a proper vertex coloring using the neighborhood of each vertex, it is natural to also study the effect of edge deletion on the sigma chromatic number of a graph and establish bounds analagous to those in Theorem 2. It is worth noting that a proper vertex coloring of a graph G induces, in different ways, both a sigma coloring and a set coloring of G; that is, $\chi(G)$ is a natural upper bound for both $\sigma(G)$ and $\chi_S(G)$.

2. Sigma Coloring and Edge Deletion

Our first result is on the bounds for $\sigma(G) - \sigma(G - e)$ for general G. The result is analogous to the result in Theorem 2.

Theorem 3. *If $e = uv$ is an edge of a graph G, then*

$$|\sigma(G) - \sigma(G - e)| \leq 2.$$

Proof. We first show that $\sigma(G - e) - \sigma(G) \leq 2$. Let c be a sigma coloring of G that uses $\sigma(G)$ colors. We will show that $G - e$ can be sigma colored using $\sigma(G) + 2$ colors. Define the coloring $\overline{c}$ on $G - e$ as follows:

$$\overline{c}(x) = \begin{cases} c(x), & x \notin \{u, v\} \\ c(x) + S, & x \in \{u, v\}, \end{cases}$$

where $S := \sum_{x \in V(G)} c(x)$. Note that $\overline{c}$ uses at most $\sigma(G) + 2$ colors. For a vertex $x \in V(G - e)$, we denote by $\overline{\sigma}(x)$ the color sum of x with respect to $\overline{c}$. Then since $\sigma(x) \leq S - c(x) < S$ for every $x \in V(G)$, we have $\overline{\sigma}(u) = \sigma(u) - c(v) < S$ and $\overline{\sigma}(v) = \sigma(v) - c(u) < S$. If y is adjacent to u or v (possibly both), then it is clear that $\overline{\sigma}(y) = \sigma(y) + S > S$ or $\overline{\sigma}(y) = \sigma(y) + 2S > S$; and so $\overline{\sigma}(y) \notin \{\overline{\sigma}(u), \overline{\sigma}(v)\}$. Now, suppose that x_1 and x_2, where both x_1 and x_2 are neither u nor v, are adjacent in $G - e$. Then exactly one of the following holds for x_1 (resp. x_2): (1) it is not adjacent to both u and v, (2) it is adjacent to u or v but not both, or (3) it is adjacent to both u and v. Thus,

$$\overline{\sigma}(x_1) \in \{\sigma(x_1), \sigma(x_1) + S, \sigma(x_1) + 2S\}$$

and

$$\overline{\sigma}(x_2) \in \{\sigma(x_2), \sigma(x_2) + S, \sigma(x_2) + 2S\}.$$

Since $\sigma(x_1) \neq \sigma(x_2)$ and by the definition of S, it follows that $\overline{\sigma}(x_1) \neq \overline{\sigma}(x_2)$. Hence, $\overline{c}$ is a sigma coloring of $G - e$ that uses at most $\sigma(G) + 2$ colors.

Now, we show that $\sigma(G) - \sigma(G - e) \leq 2$. Let c be a sigma coloring of $G - e$ that uses $\sigma(G - e)$ colors. We will show that G can be sigma colored using at most $\sigma(G - e) + 2$ colors. Note that the addition of edge e to $G - e$ (to form G) changes the color sums of only u and v. Define the coloring $\overline{c}$ on G as follows:

$$\overline{c}(x) = \begin{cases} c(x), & x \notin \{u, v\}, \\ c(x) + S, & x = u, \\ c(x) + 2S, & x = v, \end{cases}$$

where $S := \sum_{x \in V(G-e)} c(x)$. Note that $\overline{c}$ uses at most $\sigma(G - e) + 2$ colors. Again, for a vertex $x \in V(G)$, we denote by $\overline{\sigma}(x)$ the color sum of x with respect to $\overline{c}$. We have $\sigma(x) < S$ for every $x \in V(G - e)(= V(G)$. Also, $0 < \sigma(u) + c(v) \leq S$ and $0 < \sigma(v) + c(u) \leq S$ since $uv \notin E(G - e)$. It follows that

$$2S < \overline{\sigma}(u) = \sigma(u) + c(v) + 2S \leq 3S$$

and

$$S < \overline{\sigma}(v) = \sigma(v) + c(u) + S \leq 2S.$$

Thus, $\overline{\sigma}(u) \neq \overline{\sigma}(v)$.

Now, suppose y is a vertex that is neither u nor v.

- If y is adjacent to u but not to v, then $\overline{\sigma}(y) = \sigma(y) + S \leq 2S < \overline{\sigma}(u)$.
- If y is adjacent to v but not to u, then $\overline{\sigma}(y) = \sigma(y) + 2S > 2S \geq \overline{\sigma}(v)$.
- If y is adjacent to both u and v, then $\overline{\sigma}(y) = \sigma(y) + 3S$, which is clearly strictly greater than both $\overline{\sigma}(u)$ and $\overline{\sigma}(v)$.

Now, suppose x_1 and x_2, both not u nor v, are adjacent in G, then x_1 and x_2 are also adjacent in $G - e$. Similar to the previous argument, we have

$$\overline{\sigma}(x_1) \in \{\sigma(x_1), \sigma(x_1) + S, \sigma(x_1) + 2S, \sigma(x_1) + 3S\}$$

and

$$\overline{\sigma}(x_2) \in \{\sigma(x_2), \sigma(x_2) + S, \sigma(x_2) + 2S, \sigma(x_2) + 3S\}.$$

Since $\sigma(x_1) \neq \sigma(x_2)$ and by the definition of S, it follows that $\overline{\sigma}(x_1) \neq \overline{\sigma}(x_2)$. Hence, $\overline{c}$ is a sigma coloring of G that uses at most $\sigma(G) + 2$ colors. □

Example 4. *For all $m \geq 6$ and $k \in \{-1, 0\}$, there is a connected graph G, with order m, that has an edge e so that $G - e$ is connected and $\sigma(G) - \sigma(G - e) = k$.*

Proof. Consider the graph G given below.

Clearly, $\sigma(G) = 1$. Moreover, $\sigma(G - e_1) = 1$ and $\sigma(G - e_2) = 2$. □

In the above example, we considered only -1 and 0 as values for k. The case where $k = 1$ or $k = 2$ is addressed in the following. We study the existence of sequences of edge deletions each of which decreases the sigma chromatic number of a graph by one. We consider this problem for path complements, which we define as follows:

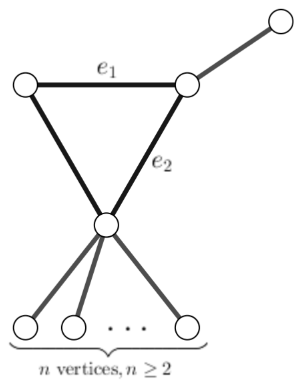

Fig. 1 The graph G with order $4 + n$.

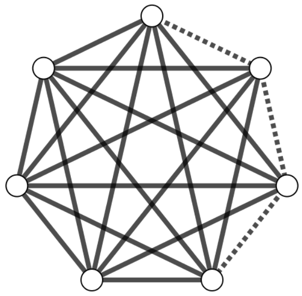

Fig. 2 The path complement $\overline{P}_{4,7}$.

Definition 5. *The* ***complement of a path*** *P_m, $m \geq 2$, in the complete graph K_n, $n \geq m$, is the graph obtained by deleting the edges of a subgraph of K_n that is isomorphic to P_m. This graph is denoted by $\overline{P}_{m,n}$.*

As an example, the graph $\overline{P}_{4,7}$ is shown in **Fig. 2** where the deleted edges are indicated using dashed segments.

Observation 6. *It is easy to see that $\overline{P}_{2,n}$, $n \geq 3$, has sigma chromatic number $n-2$; that is, deleting one edge from K_n decreases the sigma chromatic number by two.*

As a consequence of Proposition 3.1 in Ref. [4], it is worth noting that there is no sequence of edge deletions in K_n that will decrease the sigma chromatic number to $n-1$.

Our result on the sigma chromatic number of path complements is the following.

Proposition 7. *For $n \geq 4$ and $m = 2, 3, \ldots, \lceil n/2 \rceil$,*

$$\sigma(\overline{P}_{m,n}) = n - m.$$

Proof. First, note that the graph $\overline{P}_{m,n}$ has exactly one subgraph S that is isomorphic to K_{n-m}. Moreover, for each $s \in V(S)$, $N[s] = V(\overline{P}_{m,n})$. Hence, $\sigma(\overline{P}_{m,n}) \geq n - m$.

We are now left to show that $\overline{P}_{m,n}$ has a sigma coloring that uses $n-m$ colors. Let c be a sigma coloring of K_n; naturally, c uses n colors. Moreover, by setting $d = \Delta(K_n) + 1 = n$, we can choose the colors used by c to be

$$1, d, d^2, \ldots, d^{n-1}.$$

We proceed by considering the following cases.

Case 1. Suppose $n = 5$ and $m = \lceil n/2 \rceil = 3$. This case pertains to $\overline{P}_{3,5}$, for which it is easy to verify that the sigma chromatic number is $5 - 3 = 2$.

Case 2. Suppose $n \geq 7$ is odd and $m = \lceil n/2 \rceil$. Let a and b be the endvertices of the path P_m whose edges were deleted from K_n to form $\overline{P}_{m,n}$. Construct the coloring $\overline{c}$ on $\overline{P}_{m,n}$ as follows: if $x \in V(S)$, set $\overline{c}(x) = c(x)$; moreover, we define $\overline{c}$ on $V(\overline{P}_{m,n}) - V(S)$ so that

(1) $\overline{c}(V(\overline{P}_{m,n}) - V(S)) \subseteq \overline{c}(S)$,
(2) $\overline{c}(a) = \overline{c}(b)$,
(3) $\overline{c}(x) \neq \overline{c}(a)$ for all $x \in V(\overline{P}_{m,n}) - V(S)$, and
(4) $\overline{c}(x) \neq \overline{c}(y)$ for all $x, y \in V(\overline{P}_{m,n}) - V(S)$.

Note that such a coloring is possible since the vertices in $V(\overline{P}_{m,n}) - V(S)$ use only $m-1$ colors and $m - 1 = \lceil n/2 \rceil - 1 = n - m = |V(S)|$. We now show that $\overline{c}$ is a sigma coloring. Suppose x_1 and x_2 are adjacent in $\overline{P}_{m,n}$.

- Case 2.1: Suppose x_1 and x_2 are both in $V(S)$. Then $\overline{\sigma}(x_1) = \sigma(x_1)$ and $\overline{\sigma}(x_2) = \sigma(x_2)$; hence, $\overline{\sigma}(x_1) \neq \overline{\sigma}(x_2)$.
- Case 2.2: Suppose x_1 is in $V(S)$ while x_2 is in $V(\overline{P}_{m,n}) - V(S)$. Then $\deg x_1 = n - 1$ while $\deg x_2 = n - 2$. By the choice of colors of c, $\sigma(x_1) \neq \sigma(x_2)$.
- Case 2.3: Suppose $x_1 = a$ and $x_2 = b$. Then $\deg x_1 = \deg x_2 = n - 2$. Since $m \geq 4$, then x_1 and x_2 do not have the same neighbors in $V(\overline{P}_{m,n}) - V(S)$. By the construction of $\overline{c}$, $\sigma(x_1) \neq \sigma(x_2)$.
- Case 2.4: Suppose $x_1 \in \{a, b\}$ and $x_2 \in V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$. Then $\deg x_1 = n - 2$ and $\deg x_2 = n - 3$. By the choice of colors of c, $\sigma(x_1) \neq \sigma(x_2)$.
- Case 2.5: Suppose x_1 and x_2 are both in $V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$. Then $\deg x_1 = \deg x_2 = n - 3$ and $\overline{c}(x_1) \neq \overline{c}(x_2)$. Hence, $\sigma(x_1) \neq \sigma(x_2)$.

Therefore, $\overline{c}$ is a sigma coloring of $\overline{P}_{m,n}$ that uses $n - m$ colors.

Case 3. Suppose n is even or $2 \leq m \leq \lceil n/2 \rceil - 1$. Construct the coloring $\overline{c}$ on $\overline{P}_{m,n}$ as follows: if $x \in V(S)$, set $\overline{c}(x) = c(x)$; moreover, we define $\overline{c}$ on $V(\overline{P}_{m,n}) - V(S)$ so that

(1) $\overline{c}(V(\overline{P}_{m,n}) - V(S)) \subseteq \overline{c}(S)$, and
(2) $\overline{c}(x) \neq \overline{c}(y)$ for all $x, y \in V(\overline{P}_{m,n}) - V(S)$.

Note that such a coloring is possible since the vertices in $V(\overline{P}_{m,n}) - V(S)$ use only m colors and $m \leq n - m = |S|$. We now show that $\overline{c}$ is a sigma coloring. Suppose x_1 and x_2 are adjacent in $\overline{P}_{m,n}$.

- Case 3.1: Suppose x_1 and x_2 are both in $V(S)$. Then $\overline{\sigma}(x_1) = \sigma(x_1)$ and $\overline{\sigma}(x_2) = \sigma(x_2)$; hence, $\overline{\sigma}(x_1) \neq \overline{\sigma}(x_2)$.
- Case 3.2: Suppose x_1 is in $V(S)$ while x_2 is in $V(\overline{P}_{m,n}) - V(S)$. Then $\deg x_1 = n - 1$ while $\deg x_2 = n - 2$. By the choice of colors of c, $\sigma(x_1) \neq \sigma(x_2)$.
- Case 3.3: Suppose $x_1 \in \{a, b\}$ and $x_2 \in V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$. Then $\deg x_1 = n - 2$ and $\deg x_2 = n - 3$. By the choice of colors of c, $\sigma(x_1) \neq \sigma(x_2)$.
- Case 3.4: Suppose ($x_1 = a$ and $x_1 = b$) or (x_1 and x_2 are both in $V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$). Then $\deg x_1 = \deg x_2$ and $\overline{c}(x_1) \neq \overline{c}(x_2)$. Hence, $\sigma(x_1) \neq \sigma(x_2)$.

Therefore, $\overline{c}$ is a sigma coloring of $\overline{P}_{m,n}$ that uses $n - m$ colors. □

Proposition 7 implies the following: Consider a subgraph of K_n isomorphic to a path $P_m : v_1 \to v_2 \to \cdots \to v_m$, where each v_i is a vertex of K_n. The deletion of edge v_1v_2 decreases the sigma chromatic number by two. Then in the sequence of deletions of edges v_iv_{i+1} where i runs from 2 to $m - 1$, each edge deletion decreases the sigma chromatic number by one. This is illustrated for K_7 in **Fig. 3**. For comparison, the same sequence of edge deletions in Fig. 3 produces the following sequence of chromatic numbers: $\chi = 6, \chi = 6, \chi = 5$.

Example 4, Observation 6, and Proposition 7 imply the following:

Corollary 8. *For each $m \geq 6$ and for each $k \in \{-1, 0, 1, 2\}$, there is a connected graph G, with order m, that has an edge e for which $G - e$ is connected and $\sigma(G) - \sigma(G - e) = k$.*

We have not found a graph G that has an edge e for which $\sigma(G) - \sigma(G - e) = -2$. But as in Ref. [2], we have also found sufficient conditions for the inequality $\sigma(G) - \sigma(G - e) \geq -1$ to hold.

Theorem 9. *Let $e = uv$ be an edge in a graph G. If e is a bridge or $d_{G-e}(u, v) \geq 4$, then $\sigma(G) - \sigma(G - e) \geq -1$.*

Proof. Let c be a sigma coloring of G that uses $\sigma(G)$ colors. We will show that $G - e$ can be colored using $\sigma(G) + 1$ colors. Define

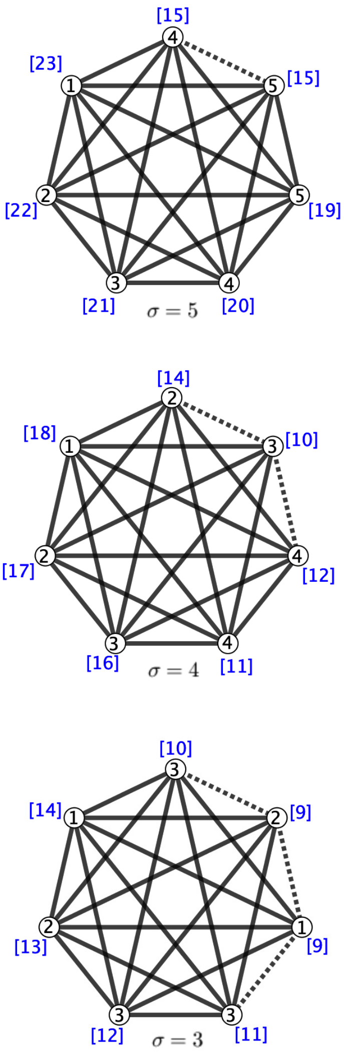

Fig. 3 A sequence of edge deletions in K_7.

$\overline{c}$ on $G - e$ as follows:

$$\overline{c}(x) = \begin{cases} S, & x \in \{u, v\}, \\ c(x), & \text{otherwise}, \end{cases}$$

where $S := \sum_{x \in V(G)} c(x)$.

Note that $\overline{c}$ uses at most $\sigma(G) + 1$ colors. We will show $\overline{c}$ is a sigma coloring of $G - e$. Let x and y be adjacent vertices in $G - e$. As detailed in Ref. [3], we can make a change of colors to ensure that $\overline{\sigma}(x) \neq \overline{\sigma}(y)$ whenever x and y are vertices of different degrees. For instance, we may first choose the colors used by c to be $1, d, d^2, \ldots, d^{\sigma(G)-1}$, where $d := \Delta(G) + 1$ and update $S := d^{\sigma(G)}$, which is greater than $\sum_{x \in V(G)} c(x)$. With this choice of colors, two adjacent vertices may have equal color sums only if they have equal degrees. Hence, we only need to consider the case that $\deg x = \deg y$.

Case 1. Suppose $x = u$. Then y cannot be adjacent to v since this will create a $u - v$ path of length 2. Also, $\sigma(y) - c(u) \geq 0$ as u and y are adjacent. In this case, $\overline{\sigma}(u) = \sigma(u) - c(v) < S$ and $\overline{\sigma}(y) = \sigma(y) - c(u) + S \geq S$. Then $\overline{\sigma}(y) \geq S > \overline{\sigma}(u)$.

Case 2. Suppose $x = v$. Then this case proceeds in a similar manner as Case 1.

We now consider the case where $\{x, y\} \cap \{u, v\} = \emptyset$. If x is adjacent to u, then x and y must not be adjacent to v since this would create a $u - v$ path of length 2 or 3. Moreover, $\sigma(x) \neq \sigma(y)$ since x and y are also adjacent in G.

Case 3. Suppose $x \in N(u)$ and $y \in N(u)$. Then $\overline{\sigma}(x) = \sigma(x) - c(u) + S \neq \sigma(y) - c(u) + S = \overline{\sigma}(y)$.

Case 4. Suppose $x \in N(u)$ and $y \notin N(u)$. Then $\overline{\sigma}(x) = \sigma(x) - c(u) + S \neq \sigma(y) = \overline{\sigma}(y)$.

Case 5. Suppose $x \notin N(u)$ and $y \notin N(u)$. Then $\overline{\sigma}(x) = \sigma(x) \neq \sigma(y) = \overline{\sigma}(y)$.

Therefore, $\overline{c}$ is a sigma coloring of $G - e$ that uses $\sigma(G) + 1$ colors. □

In the following, we consider edge deletions in regular graphs of order at least 2.

Proposition 10. *Suppose G is a connected regular graph with order at least 2.*

(1) *For any edge $e = uv$ in G, $\sigma(G - e) \leq \sigma(G)$.*

(2) *If G is not complete and $e = uv \notin E(G)$, then $\sigma(G + e) \leq \sigma(G) + 1$.*

Proof. (1) Suppose c is a sigma coloring of G that uses $\sigma(G)$ colors. Let $\overline{c}$ be the coloring of $G - e$ so that $\overline{c}(x) = c(x)$ for each $x \in V(G - e) = V(G)$. We show that $\overline{c}$ is a sigma coloring of $G - e$. First, $\overline{\sigma}(x) = \sigma(x)$ for each $x \notin \{u, v\}$. Let x and y be adjacent vertices in $G - e$. If they have different degrees, then $\overline{\sigma}(x) \neq \overline{\sigma}(y)$ (possibly needing a change of colors as in the proof of Theorem 9). If they have equal degrees, then $\overline{\sigma}(x) = \sigma(x) \neq \sigma(y) = \overline{\sigma}(y)$.

(2) Let c be a sigma coloring of G that uses $\sigma(G)$ colors. Let $\overline{c}$ be the coloring of $G + e$ where $\overline{c}(x) = c(x)$ if $x \neq v$ and $\overline{c}(v) = S := \sum_{z \in V(G)} c(z)$. Let x, y be adjacent vertices of $G + e$ with equal degrees. Then $\{x, y\} = \{u, v\}$ or $\{x, y\} \cap \{u, v\} = \emptyset$.

(a) If x and y are both not in $N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) \neq \sigma(y) = \overline{\sigma}(y)$;

(b) If x and y are both in $N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) - c(v) + S \neq \sigma(y) - c(v) + S = \overline{\sigma}(y)$;

(c) If exactly one of x and y is in $N_G(v)$, say $x \in N_G(v)$ and $y \notin N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) - c(v) + S > \sigma(y) = \overline{\sigma}(y)$. This also covers the case where $\{x, y\} = \{u, v\}$.

□

3. On the Sigma Chromatic Number of Complements of Paths and Cycles

In this section, we determine a lower bound for the sigma chromatic number of the complement of a cycle or a path. For convenience, we introduce the following notations. For a cycle $C_n = v_1 v_2 \cdots v_n v_1$, $n \geq 3$ and for each $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, we denote by A_k the triple of vertices $(v_{2k-1}, v_{2k}, v_{2k+1})$ and by B_k the triple of vertices $(v_{2k-2}, v_{2k-1}, v_{2k})$ (Note that the subscripts are computed modulo n). For example, in $C_7 = v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_1$, we have

$$A_1 = (v_1, v_2, v_3), \quad A_2 = (v_3, v_4, v_5), \quad A_3 = (v_5, v_6, v_7),$$

and

$$B_1 = (v_7, v_1, v_2), \quad B_2 = (v_2, v_3, v_4), \quad B_3 = (v_4, v_5, v_6).$$

Given an ordered triple T of vertices (e.g., some A_k or B_k) and a vertex coloring c of C_n or $\overline{C_n}$, we denote by $c(T)$ the multiset of colors used in the vertices in T. Note that $c(T)$ is a multiset and not an ordered triple. The following is an important observation.

Observation 11. *If c is a sigma coloring of $\overline{C_n}$, then for any triple T and T' of consecutive vertices in C_n, we must have $c(T) \neq c(T')$ if $|T \cap T'| \leq 1$. In particular, for any distinct k, j, we must have $c(A_k) \neq c(A_j)$ and $c(B_k) \neq c(B_j)$.*

The above observation follows from the fact that if v is the middle vertex in a triple T, then $\sigma(v) = S - \sum_{x \in T} c(x)$, where $S := \sum_{z \in V(\overline{C_n})} c(z)$.

Proposition 12. *Let m be a positive integer and set $M = \binom{m+2}{3}$. Then $\sigma(\overline{C_n}) > m$ for all $n \geq 2M + 1$.*

Proof. Suppose c is a vertex coloring of $\overline{C_n}$ that uses m colors. Moreover, assume that the colors are $1, d, d^2, \ldots, d^{m-1}$, where $d = n - 2$. Then the number of 3-multisets that can be formed using these m colors (repetition of colors allowed) is M. By the choice of colors, it also follows that there are M possible color sums.

Suppose $n \geq 2M + 2$. Then $\lfloor \frac{n}{2} \rfloor > \frac{n}{2} - 1 \geq M$. By Observation 11, we must have $M \geq \lfloor \frac{n}{2} \rfloor$. Therefore, c is not a sigma coloring of $\overline{C_n}$ and $\sigma(\overline{C_n}) > m$.

Now, suppose $n = 2M + 1$. Then $\lfloor n/2 \rfloor = M$. For c to be a sigma coloring, by Observation 11, $c(A_1), c(A_2), \ldots, c(A_M)$ must be distinct triples. Furthermore, $c(B_1)$ must be distinct from $c(A_2), c(A_3), \ldots, c(A_M)$. Then $c(B_1) = c(A_1)$. Similarly, $c(B_2)$ must be distinct from $c(A_3), c(A_4), \ldots, c(A_M)$ and $c(B_1) = c(A_1)$; thus, $c(B_2) = c(A_2)$. Proceeding in this manner, we conclude that we must have $c(A_k) = c(B_k)$ for all $k = 1, 2, \ldots, M$. Now, consider the triple $T = (v_{2M}, v_{2M+1}, v_1)$. Again, for c to be a sigma coloring, we must have $c(T)$ distinct from $c(A_1), c(A_2), \ldots, c(A_{M-1})$ and $c(B_M) = c(A_M)$. But since T is a triple not in $\{A_k, B_k : k = 1, 2, \ldots, M\}$, $c(T)$ will have to be one of $c(A_1), c(A_2), \ldots, c(A_{M-1}), c(A_M)$, which implies that c is not a sigma coloring of $\overline{C_n}$. Therefore, $\sigma(\overline{C_n}) > m$. □

We now turn to the complements of paths. Suppose $P_n = v_1 v_2 \cdots v_n$, $n \geq 3$. Note that the vertices $v_2, v_3, \ldots, v_{n-1}$, which are of degree $n-3$ in $\overline{P_n}$, will also have color sums corresponding to 3-multisets of colors. Hence, by arguing in a similar manner as in Proposition 12, we obtain the following.

Proposition 13. *Let m be a positive integer and set $M = \binom{m+2}{3}$. Then $\sigma(\overline{P_n}) > m$ for all $n \geq 2M + 3$.*

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