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CHARACTERIZING CONVERGENCE CONDITIONS FOR THE M_α-INTEGRAL

Ian June Luzon Garces* and Abraham Perral Racca**

Abstract. Park, Ryu, and Lee recently defined a Henstock-type integral, which lies entirely between the McShane and the Henstock integrals. This paper presents two characterizing convergence conditions for this integral, and derives other known convergence theorems as corollaries.

1. Introduction

Park, Ryu, and Lee [2] recently defined a Henstock-type integral, and they call it M_α-integral. Several properties of M_α-integral were established in [2] and [3]. Most of them parallel the usual properties of an integral. One of these results is the Saks-Henstock Lemma. Moreover, by providing examples, it was also shown that M_α-integral lies strictly between the McShane and the Henstock integrals.

Let $\alpha > 0$ be a constant, and $I = [a, b]$ a non-degenerate closed and bounded interval in $\mathbb{R}$. The following terms and notations are from [2].

(1) A *partial partition* D of I is a finite collection of interval-point pairs $([u, v], \xi)$ such that the closed intervals $[u, v]$ are non-overlapping, $\bigcup[u, v] \subseteq I$, and $\xi \in I$. If $\bigcup[u, v] = I$, we call the partition D simply a *partition*.

(2) A positive function defined on I is called a *gauge* on I.

(3) Let δ be a gauge on I, and $D = \{([u, v], \xi)\}$ a partial partition of I. If $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$ for all $([u, v], \xi) \in D$, then we say that D is a *δ-fine McShane partial partition.* Moreover, if D is a McShane partial partition such that $\xi \in [u, v]$ for all $([u, v], \xi) \in D$, then D is called a *δ-fine Henstock partial partition.*

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(4) A McShane partition $D = \{([u,v],\xi)\}$ of I is said to be an M_α-*partition* if

$$\sum_{([u,v],\xi)\in D} \text{dist}(\xi,[u,v]) < \alpha,$$

where $\text{dist}(x,J) = \inf\{|y-x| : y \in J\}$.

(5) Let $D = \{([u,v],\xi)\}$ be a partial partition on I, and f a real-valued function defined on I. We write

$$S(f,D) = \sum_{([u,v],\xi)\in D} f(\xi)(v-u).$$

With these terms and notations, the definition of M_α-integrability can now be presented.

Definition 1.1 ([2, Definition 2.1]). A function $f : I \to \mathbb{R}$ is M_α-*integrable* if there exists a real number A such that, for each $\epsilon > 0$, there is a gauge δ on I such that

$$|S(f,D) - A| < \epsilon$$

for each δ-fine M_α-partition D of I. Here, A is called the M_α*-integral* of f on I, and we write $A = \int_I f$.

This paper presents two characterizing convergence conditions for this new integral, and derives other known convergence theorems as corollaries. The paper is outlined as follows. The main theorem and its corollaries are presented in Section 2, while the proof of the main theorem is shown in Section 3. The importance of one characterizing convergence condition is shown by an example in Section 4.

Throughout the discussion, given a set $E \subseteq \mathbb{R}$, we denote E^c its complement and $\mu(E)$ its Lebesgue outer measure.

2. Main theorem and its consequences

Let $\{f_n\}$ be a sequence of M_α-integrable functions on $I = [a,b]$, and $f_n(x) \to f(x)$ for all $x \in I$. We say that $\{f_n\}$ is

(1) M_α*-convergent in Gordon's sense* if for every $\epsilon > 0$ there is a gauge δ on I such that, for each δ-fine M_α-partition D of I, there corresponds an integer $N_D > 0$ with the following property:

$$\left| S(f_n,D) - \int_I f_n \right| < \epsilon \qquad \text{for all } n \geq N_D.$$

(2) *M_α-convergent to f in Bartle's sense* if for every $\epsilon > 0$ there corresponds an integer $N_\epsilon > 0$ such that if $n \geq N_\epsilon$ there is gauge δ_n on I such that, for each δ_n-fine M_α-partition D of I,

$$|S(f_n, D) - S(f, D)| < \epsilon.$$

(3) *equi-integrable* if for every $\epsilon > 0$ there is a gauge δ on I such that if D is any δ-fine M_α-partition of I, then

$$\left| S(f_n, D) - \int_I f_n \right| < \epsilon \qquad \text{for all } n.$$

(3) *dominated* if there are M_α-integrable functions g and h on I such that $g(x) \leq f_n(x) \leq h(x)$ for all $x \in I$ and for all n.

(4) *monotone* if either $f_n(x) \leq f_{n+1}(x)$ for all $x \in I$ and for all n, or $f_n(x) \geq f_{n+1}(x)$ for all $x \in I$ and for all n.

We now present the main theorem of the paper, whose proof is postponed to the next section.

THEOREM 2.1. *Let $\{f_n\}$ be a sequence of M_α-integrable functions on I such that $f_n(x) \to f(x)$ for all $x \in I$. Then the following statements are equivalent:*

(i) *f is M_α-integrable on I, and*

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

(ii) *$\{f_n\}$ is M_α-convergent in Gordon's sense on I.*

(iii) *$\{f_n\}$ is M_α-convergent to f in Bartle's sense on I.*

Observe that an equi-integrable sequence is a special type of M_α-convergent sequence in Gordon's sense. Thus, the following corollary is an immediate consequence of our main theorem.

COROLLARY 2.2 (Equi-integrability). *Let $\{f_n\}$ be a sequence of M_α-integrable functions on I such that $f_n(x) \to f(x)$ for all $x \in I$. If $\{f_n\}$ is an equi-integrable sequence, then f is M_α-integrable on I, and*

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

We now prove that a dominated sequence and a monotone sequence are special types of equi-integrable sequence. Following [1, Theorem 9.13(b)], every nonnegative M_α-integrable function on I is also McShane integrable there, and their integrals are equal. We use this fact in the proofs of the following corollaries.

Corollary 2.3 (Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of M_α-integrable functions on I such that $f_n(x) \to f(x)$ for all $x \in I$. If $\{f_n\}$ is a dominated sequence, then it is equi-integrable on I. Consequently, the function f is M_α-integrable on I, and*

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Proof. It is not difficult to verify that $|f_n - f_m| \le h - g$ on I and for all m, n. Let $\epsilon > 0$ be given.

The function $\varphi = h - g$ is M_α-integrable function on I, and so there exists a gauge δ_φ on I such that if D is a δ_φ-fine M_α-partition of I, then

$$\left| S(\varphi, D) - \int_I \varphi \right| < \epsilon.$$

Since it is nonnegative, the function φ is also McShane integrable on I, which implies that the function $\Phi(x) = \int_a^x \varphi$ is absolutely continuous on I; that is, there exists a number $\eta > 0$ such that if $\{[a_i, b_i] : i = 1, 2, \ldots, m\}$ is a finite collection of closed intervals in $[a, b]$ with $\sum_{i=1}^m (b_i - a_i) < \eta$, then

$$\sum_{i=1}^m |\Phi(b_i) - \Phi(a_i)| < \epsilon.$$

Furthermore, by Egorov's Theorem [1, Theorem 2.13], there exists an open set $O \subset I$ such that $\mu(O) < \eta$ and f_n converges uniformly to f on $I \setminus O$. Choose an integer $N > 0$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $m, n \ge N$ and for all $x \in I \setminus O$.

Define a gauge δ_1 on I as follows:

$$\delta_1(x) = \begin{cases} \delta_\varphi(x) & \text{if } x \in I \setminus O \\ \min\{\delta_\varphi(x), \operatorname{dist}(x, O^c)\} & \text{if } x \in O. \end{cases}$$

Consider a δ_1-fine M_α-partition D of I, and integers $m, n \ge N$. Let D_1 be the subset of D that has tags in $I \setminus O$, and let $D_2 = D \setminus D_1$. Also, let $I_1 = \bigcup\{[u, v] : ([u, v], \xi) \in D_1\}$ and $I_2 = \bigcup\{[u, v] : ([u, v], \xi) \in D_2\}$. Using Saks-Henstock Lemma [2, Lemma 2.5] and the fact that $\mu(I_2) < \eta$, we obtain

$$\begin{aligned}
&|S(f_n,D)-S(f_m,D)| \\
&\quad\le\ |S(f_n,D_1)-S(f_m,D_1)|+|S(f_m,D_2)-S(f_m,D_2)| \\
&\quad<\ \epsilon\mu(I_1)+S(\varphi,D_2) \\
&\quad\le\ \epsilon(b-a)+\left|S(\varphi,D_2)-\int_{I_2}\varphi\right|+\sum_{([u,v],\xi)\in D_2}|\Phi(v)-\Phi(u)| \\
&\quad<\ \epsilon(b-a)+2\epsilon
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_I f_n-\int_I f_m\right| \ &\le\ \left|\int_{I_1} f_n-S(f_n,D_1)\right|+\left|\int_{I_2} f_n-S(f_n,D_2)\right| \\
&\quad+\left|\int_{I_1} f_m-S(f_m,D_1)\right|+\left|\int_{I_2} f_m-S(f_m,D_2)\right| \\
&\quad+|S(f_n,D)-S(f_m,D)| \\
&<\ 6\epsilon+\epsilon(b-a).
\end{aligned}$$

Since each f_n is M_α-integrable on I, there exists a gauge $\delta\le\delta_1$ on I such that if D is any δ-fine M_α-partition of I, then

$$\left|S(f_n,D)-\int_I f_n\right|<\epsilon$$

for $1\le n\le N$, and

$$\begin{aligned}
&\left|S(f_n,D)-\int_I f_n\right| \\
&\le|S(f_n,D)-S(f_N,D)|+\left|S(f_N,D)-\int_I f_N\right|+\left|\int_I f_N-\int_I f_n\right| \\
&<9\epsilon+2\epsilon(b-a)
\end{aligned}$$

for $n>N$. Therefore, the sequence $\{f_n\}$ is equi-integrable on I, and the rest of the corollary follows from Corollary 2.2. □

Corollary 2.4 (Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of M_α-integrable functions on I such that $f_n(x)\to f(x)$ for all $x\in I$. If $\{f_n\}$ is a monotone sequence and $\lim_{n\to\infty}\int_I f_n<\infty$, then $\{f_n\}$ is equi-integrable on I. Consequently, the function f is M_α-integrable on I, and*

$$\int_I f=\lim_{n\to\infty}\int_I f_n.$$

Proof. Assume that $\{f_n\}$ is nondecreasing. The proof for the case when $\{f_n\}$ is non-increasing is similar. Then, for each n, the function $f_n - f_1$ is nonnegative and M_α-integrable on I. It follows that each function $f_n - f_1$ is McShane integrable on I. Since $(f_n - f_1) \nearrow (f - f_1)$ and $\lim\limits_{n\to\infty} \int_I (f_n - f_1) < \infty$, by the Monotone Convergence Theorem for McShane Integral [1, Corollary 13.4], the function $f - f_1$ is McShane integrable (and so M_α-integrable) on I. Thus, the function f is M_α-integrable on I. Since $f_1(x) \le f_n(x) \le f(x)$ for all $x \in I$ and for all n, applying Corollary 2.3 completes the proof. □

3. Proof of the main theorem

We will use the following lemma.

LEMMA 3.1. *Let $\{f_n\}$ be a sequence of M_α-integrable functions on I such that $f_n(x) \to f(x)$ for all $x \in I$. If $\{f_n\}$ is M_α-convergent in Gordon's or Bartle's sense, then $\{\int_I f_n\}$ is a Cauchy sequence in $\mathbb{R}$.*

Proof. Suppose $\{f_n\}$ is M_α-convergent in Gordon's sense on I. Let $\epsilon > 0$ be given. Then there exists a gauge δ on I, and we can choose a particular δ-fine M_α-partition D_0 of I such that

$$\left| S(f_n, D_0) - \int_I f_n \right| < \epsilon \qquad \text{for all } n \ge N_{D_0}$$

for some positive integer N_{D_0}, as guaranteed in the definition of M_α-convergent sequence in Gordon's sense. Since D_0 is finite and $f_n(x) \to f(x)$ for all $x \in I$, there exists an integer $N \ge N_{D_0}$ such that

$$|S(f_n, D_0) - S(f_m, D_0)| < \epsilon \qquad \text{for all } m, n \ge N,$$

which implies that

$$\begin{aligned}
&\left| \int_I f_n - \int_I f_m \right| \\
&\le \left| \int_I f_n - S(f_n, D_0) \right| + |S(f_n, D_0) - S(f_m, D_0)| + \left| S(f_m, D_0) - \int_I f_m \right| \\
&< 3\epsilon.
\end{aligned}$$

Thus, $\left\{\int_I f_n\right\}$ is a Cauchy sequence in $\mathbb{R}$.

On the other hand, suppose $\{f_n\}$ is M_α-convergent to f in Bartle's sense on I. Let $\epsilon > 0$ be given, and let N_ϵ be the positive integer guaranteed by the definition of M_α-convergence in Bartle's sense. If m

and n are integers such that $m, n \geq N_\epsilon$, then there exist gauges δ_m and δ_n on I such that, for every δ_m-fine M_α-partition D_m of I, we have

$$|S(f_m, D_m) - S(f, D_m)| < \epsilon,$$

and, for every δ_n-fine M_α-partition D_n of I, we have

$$|S(f_n, D_n) - S(f, D_n)| < \epsilon.$$

Furthermore, since f_m and f_n are M_α-integrable on I, there exist gauges δ'_m, δ'_n on I such that, for every δ'_m-fine M_α-partition D'_m of I,

$$\left| S(f_m, D'_m) - \int_I f_m \right| < \epsilon$$

and, for every δ'_n-fine M_α-partition D'_n of I,

$$\left| S(f_n, D'_n) - \int_I f_n \right| < \epsilon.$$

Set $\delta_\epsilon(x) = \min\{\delta_m(x), \delta_n(x), \delta'_m(x), \delta'_n(x)\}$ for $x \in I$. Therefore, if D is δ_ϵ-fine M_α-partition of I, then

$$\begin{aligned} \left| \int_I f_n - \int_I f_m \right| &\leq \left| \int_I f_n - S(f_n, D) \right| + |S(f_n, D) - S(f, D)| \\ &\quad + |S(f, D) - S(f_m, D)| + \left| S(f_m, D) - \int_I f_m \right| \\ &< 4\epsilon, \end{aligned}$$

and so $\left\{\int_I f_n\right\}$ is a Cauchy sequence in $\mathbb{R}$. □

We are now ready to prove the Main Theorem (Theorem 2.1). We break the proof into four parts.

(i) $\Longrightarrow$ (ii). Suppose that f is M_α-integrable on I, and

$$\int_I f = \lim_{n \to \infty} \int_I f_n.$$

Let $\epsilon > 0$ be given. Then there is an integer $N > 0$ such that

$$\left| \int_I f - \int_I f_n \right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq N,$$

and there exists a gauge δ on I such that if D is a δ-fine M_α-partition of I, then

$$\left| S(f, D) - \int_I f \right| < \frac{\epsilon}{3}.$$

Since each such D is finite and since $f_n(x) \to f(x)$ for all $x \in I$, we can choose for each D an integer $N_D \geq N$ such that

$$|S(f_n, D) - S(f, D)| < \frac{\epsilon}{3} \qquad \text{for all } n \geq N_D.$$

Thus, for every integer $n \geq N_D$, we have

$$\begin{aligned}&\left|S(f_n, D) - \int_I f_n\right| \\ &\leq |S(f_n, D) - S(f, D)| + \left|S(f, D) - \int_I f\right| + \left|\int_I f - \int_I f_n\right| \\ &< \epsilon,\end{aligned}$$

and so $\{f_n\}$ is M_α-convergent in Gordon's sense on I.

(ii) $\Longrightarrow$ (i). Suppose that $\{f_n\}$ is M_α-convergent in Gordon's sense on I. By Lemma 3.1, the sequence $\left\{\int_I f_n\right\}$ is Cauchy in $\mathbb{R}$, and so it converges to some real number A. Let $\epsilon > 0$ be given. Then there exists an integer $N > 0$ such that

$$\left|\int_I f_n - A\right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq N,$$

and, as guaranteed in the definition of M_α-convergence in Gordon's sense, there exists a gauge δ on I such that if D is a δ-fine M_α-partition of I, then

$$\left|\int_I f_n - S(f_n, D)\right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq N_D$$

for some integer $N_D \geq N$. Since each such D is finite and since $f_n(x) \to f(x)$ for all $x \in I$, there is an integer $n_1 \geq N_D$ such that

$$|S(f, D) - S(f_{n_1}, D)| < \frac{\epsilon}{3}.$$

Therefore, if D is any δ-fine M_α-partition of I, then

$$\begin{aligned}&|S(f, D) - A| \\ &\leq |S(f, D) - S(f_{n_1}, D)| + \left|S(f_{n_1}, D) - \int_I f_{n_1}\right| + \left|\int_I f_{n_1} - A\right| \\ &< \epsilon,\end{aligned}$$

which implies that f is M_α-integrable to A on I, and

$$\int_I f = A = \lim_{n\to\infty} \int_I f_n.$$

(i) $\Longrightarrow$ (iii). Suppose that f is M_α-integrable on I, and

$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$

Let $\epsilon > 0$ be given. Then there is an integer $N > 0$ such that

$$\left|\int_I f - \int_I f_n\right| < \frac{\epsilon}{3} \qquad \text{for all } n \geq N.$$

Since f_n is M_α-integrable on I, there is a gauge γ_n on I such that, for every γ_n-fine M_α-partition D_n of I,

$$\left|S(f_n, D_n) - \int_I f_n\right| < \frac{\epsilon}{3}.$$

Since f is M_α-integrable on I, there is a gauge δ on I such that, for every δ-fine M_α-partition D of I,

$$\left|S(f, D) - \int_I f\right| < \frac{\epsilon}{3}.$$

Let $\delta_n = \min\{\delta, \gamma_n\}$. For $n \geq N$, if D is δ_n-fine M_α-partition of I, then

$$\begin{aligned} |S(f_n, D) - S(f, D)| &\leq \left|S(f_n, D) - \int_I f_n\right| + \left|\int_I f_n - \int_I f\right| \\ &\quad + \left|\int_I f - S(f, D)\right| \\ &< \epsilon, \end{aligned}$$

which implies that $\{f_n\}$ is M_α-convergent to f in Bartle's sense on I.

(iii) $\Longrightarrow$ (i). Suppose that $\{f_n\}$ is M_α-convergent to f in Bartle's sense on I. By Lemma 3.1, the sequence $\left\{\int_I f_n\right\}$ is Cauchy in $\mathbb{R}$, and so it converges to some real number A. Let $\epsilon > 0$ be given. Then there exists an integer $N_\epsilon > 0$ such that

$$\left|\int_I f_{N_\epsilon} - A\right| < \frac{\epsilon}{3},$$

and there is a gauge δ_{N_ϵ} on I such that, for every δ_{N_ϵ}-fine M_α-partition D of I, we have

$$|S(f, D) - S(f_{N_\epsilon}, D)| < \frac{\epsilon}{3}.$$

Moreover, since f_{N_ϵ} is M_α-integrable on I, there is a gauge γ_{N_ϵ} on I such that, for every γ_{N_ϵ}-fine M_α-partition D of I,

$$\left|S(f_{N_\epsilon}, D) - \int_I f_{N_\epsilon}\right| < \frac{\epsilon}{3}.$$

Let $\delta_\epsilon(\cdot) = \min\{\delta_{N_\epsilon}(\cdot), \gamma_{N_\epsilon}(\cdot)\}$. If D is δ_ϵ-fine M_α-partition of I, then

$$\begin{aligned}&|S(f,D) - A|\\ &\le |S(f,D) - S(f_{N_\epsilon}, D)| + \left|S(f_{N_\epsilon}, D) - \int_I f_{N_\epsilon}\right| + \left|\int_I f_{N_\epsilon} - A\right|\\ &< \epsilon,\end{aligned}$$

which implies that f is M_α-integrable to A on I, and

$$\int_I f = A = \lim_{n\to\infty} \int_I f_n.$$

This ends the proof of the theorem. □

4. An example

To exhibit the importance of the main theorem, we now give a sequence of M_α-integrable functions that is not equi-integrable but is M_α-convergent to a function in Bartle's sense.

For each positive integer n, define

$$f_n(x) = \begin{cases} n & \text{if } x \in (1/n, 2/n),\\ -n & \text{if } x \in (2/n, 3/n),\\ 0 & \text{if } x \in [0,3] \setminus \{(1/n, 2/n) \cup (2/n, 3/n)\}.\end{cases}$$

It is not difficult to compute that, for each n, f_n is M_α-integrable to 0 on $[0,3]$.

Claim 1. $\{f_n\}$ is M_α-convergent to 0 in Bartle's sense.

To see this, it is not difficult to observe that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for each } x \in [0,3].$$

Let $\epsilon > 0$, and choose N_ϵ to be a positive integer such that $4N_\epsilon^{-1} < \epsilon$. For every positive integer $n \ge N_\epsilon$, we define a gauge on I as follows:

$$\delta_n(x) = \begin{cases} \frac{1}{2}\text{dist}(x, \{1/n, 2/n, 3/n\}) & \text{if } x \notin \{1/n, 2/n, 3/n\},\\ n^{-2} & \text{otherwise.}\end{cases}$$

If $D = \{(I,t)\}$ be δ_n-fine M_α-partition of I, we have

$$|S(f_n, D)| \le \frac{4}{n} \le \frac{4}{N_\epsilon} < \epsilon,$$

which ends the proof of Claim 1.

CLAIM 2. Let $0 < \epsilon < 1$. For any gauge δ on $[0,3]$, there is a δ-fine partial M_α-partition $\hat{D} = \{([u,v],\xi)\}$ of $[0,3]$ such that, for some positive integer n,

$$\left| S(f_n, \hat{D}) - \int_{\bigcup [u,v]} f_n \right| > \epsilon.$$

To prove this claim, let δ be an arbitrary gauge on $[0,3]$, and choose a positive integer n such that $2n^{-1} < \delta(0)$. Then $\hat{D} = \{([0,2n^{-1}],0)\}$ is δ-fine partial M_α-partition of $[0,3]$, and

$$\left| S(f_n, \hat{D}) - \int_{[0,2n^{-1}]} f_n \right| = |f_n(0)(2/n) - 1| = |0-1| = 1 > \epsilon,$$

which ends the proof of the second claim.

CLAIM 3. $\{f_n\}$ is not equi-integrable.

To prove this, we suppose that $\{f_n\}$ is equi-integrable. Let $0 < \epsilon < 1$ be given. Then there is a gauge δ on $[0,3]$ such that for any δ-fine M_α-partition D of $[0,3]$

$$\left| S(f_n, D) - \int_{[0,3]} f_n \right| < \epsilon \text{ for all positive integer } n.$$

By Saks-Henstock Lemma [2, Lemma 2.5], for any δ-fine partial M_α-partition $\hat{D} = \{(I_i, t_i)\}$ of $[0,3]$, we get

$$\left| S(f_n, \hat{D}) - \int_{\cup I_i} f_n \right| < \epsilon \quad \text{for all positive integer } n,$$

a contradiction to Claim 2. □

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