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## CHARACTERIZING CONVERGENCE CONDITIONS FOR THE $M_\alpha$ -INTEGRAL

IAN JUNE LUZON GARCES\* AND ABRAHAM PERRAL RACCA\*\*

ABSTRACT. Park, Ryu, and Lee recently defined a Henstock-type integral, which lies entirely between the McShane and the Henstock integrals. This paper presents two characterizing convergence conditions for this integral, and derives other known convergence theorems as corollaries.

### 1. Introduction

Park, Ryu, and Lee [2] recently defined a Henstock-type integral, and they call it  $M_\alpha$ -integral. Several properties of  $M_\alpha$ -integral were established in [2] and [3]. Most of them parallel the usual properties of an integral. One of these results is the Saks-Henstock Lemma. Moreover, by providing examples, it was also shown that  $M_\alpha$ -integral lies strictly between the McShane and the Henstock integrals.

Let  $\alpha > 0$  be a constant, and  $I = [a, b]$  a non-degenerate closed and bounded interval in  $\mathbb{R}$ . The following terms and notations are from [2].

- (1) A *partial partition*  $D$  of  $I$  is a finite collection of interval-point pairs  $([u, v], \xi)$  such that the closed intervals  $[u, v]$  are non-overlapping,  $\bigcup [u, v] \subseteq I$ , and  $\xi \in I$ . If  $\bigcup [u, v] = I$ , we call the partition  $D$  simply a *partition*.
- (2) A positive function defined on  $I$  is called a *gauge* on  $I$ .
- (3) Let  $\delta$  be a gauge on  $I$ , and  $D = \{([u, v], \xi)\}$  a partial partition of  $I$ . If  $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$  for all  $([u, v], \xi) \in D$ , then we say that  $D$  is a  *$\delta$ -fine McShane partial partition*. Moreover, if  $D$  is a McShane partial partition such that  $\xi \in [u, v]$  for all  $([u, v], \xi) \in D$ , then  $D$  is called a  *$\delta$ -fine Henstock partial partition*.

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Received April 12, 2011; Accepted August 25, 2011.

2010 Mathematics Subject Classification: Primary 26A39.

Key words and phrases:  $M_\alpha$ -integral,  $M_\alpha$ -convergence, equi-integrability.

\*This work by this author is partially supported by The Hongkong & Shanghai Banking Corporation Endowed Fund.

- (4) A McShane partition  $D = \{([u, v], \xi)\}$  of  $I$  is said to be an  $M_\alpha$ -partition if

$$\sum_{([u, v], \xi) \in D} \text{dist}(\xi, [u, v]) < \alpha,$$

where  $\text{dist}(x, J) = \inf\{|y - x| : y \in J\}$ .

- (5) Let  $D = \{([u, v], \xi)\}$  be a partial partition on  $I$ , and  $f$  a real-valued function defined on  $I$ . We write

$$S(f, D) = \sum_{([u, v], \xi) \in D} f(\xi)(v - u).$$

With these terms and notations, the definition of  $M_\alpha$ -integrability can now be presented.

DEFINITION 1.1 ([2, Definition 2.1]). A function  $f : I \rightarrow \mathbb{R}$  is  $M_\alpha$ -integrable if there exists a real number  $A$  such that, for each  $\epsilon > 0$ , there is a gauge  $\delta$  on  $I$  such that

$$|S(f, D) - A| < \epsilon$$

for each  $\delta$ -fine  $M_\alpha$ -partition  $D$  of  $I$ . Here,  $A$  is called the  $M_\alpha$ -integral of  $f$  on  $I$ , and we write  $A = \int_I f$ .

This paper presents two characterizing convergence conditions for this new integral, and derives other known convergence theorems as corollaries. The paper is outlined as follows. The main theorem and its corollaries are presented in Section 2, while the proof of the main theorem is shown in Section 3. The importance of one characterizing convergence condition is shown by an example in Section 4.

Throughout the discussion, given a set  $E \subseteq \mathbb{R}$ , we denote  $E^c$  its complement and  $\mu(E)$  its Lebesgue outer measure.

## 2. Main theorem and its consequences

Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I = [a, b]$ , and  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . We say that  $\{f_n\}$  is

- (1)  $M_\alpha$ -convergent in Gordon's sense if for every  $\epsilon > 0$  there is a gauge  $\delta$  on  $I$  such that, for each  $\delta$ -fine  $M_\alpha$ -partition  $D$  of  $I$ , there corresponds an integer  $N_D > 0$  with the following property:

$$\left| S(f_n, D) - \int_I f_n \right| < \epsilon \quad \text{for all } n \geq N_D.$$

- (2)  $M_\alpha$ -convergent to  $f$  in Bartle's sense if for every  $\epsilon > 0$  there corresponds an integer  $N_\epsilon > 0$  such that if  $n \geq N_\epsilon$  there is gauge  $\delta_n$  on  $I$  such that, for each  $\delta_n$ -fine  $M_\alpha$ -partition  $D$  of  $I$ ,

$$|S(f_n, D) - S(f, D)| < \epsilon.$$

- (3) *equi-integrable* if for every  $\epsilon > 0$  there is a gauge  $\delta$  on  $I$  such that if  $D$  is any  $\delta$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\left| S(f_n, D) - \int_I f_n \right| < \epsilon \quad \text{for all } n.$$

- (3) *dominated* if there are  $M_\alpha$ -integrable functions  $g$  and  $h$  on  $I$  such that  $g(x) \leq f_n(x) \leq h(x)$  for all  $x \in I$  and for all  $n$ .

- (4) *monotone* if either  $f_n(x) \leq f_{n+1}(x)$  for all  $x \in I$  and for all  $n$ , or  $f_n(x) \geq f_{n+1}(x)$  for all  $x \in I$  and for all  $n$ .

We now present the main theorem of the paper, whose proof is postponed to the next section.

**THEOREM 2.1.** *Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . Then the following statements are equivalent:*

- (i)  $f$  is  $M_\alpha$ -integrable on  $I$ , and

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

- (ii)  $\{f_n\}$  is  $M_\alpha$ -convergent in Gordon's sense on  $I$ .  
 (iii)  $\{f_n\}$  is  $M_\alpha$ -convergent to  $f$  in Bartle's sense on  $I$ .

Observe that an equi-integrable sequence is a special type of  $M_\alpha$ -convergent sequence in Gordon's sense. Thus, the following corollary is an immediate consequence of our main theorem.

**COROLLARY 2.2 (Equi-integrability).** *Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . If  $\{f_n\}$  is an equi-integrable sequence, then  $f$  is  $M_\alpha$ -integrable on  $I$ , and*

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

We now prove that a dominated sequence and a monotone sequence are special types of equi-integrable sequence. Following [1, Theorem 9.13(b)], every nonnegative  $M_\alpha$ -integrable function on  $I$  is also McShane integrable there, and their integrals are equal. We use this fact in the proofs of the following corollaries.

**COROLLARY 2.3** (Dominated Convergence Theorem). *Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . If  $\{f_n\}$  is a dominated sequence, then it is equi-integrable on  $I$ . Consequently, the function  $f$  is  $M_\alpha$ -integrable on  $I$ , and*

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

*Proof.* It is not difficult to verify that  $|f_n - f_m| \leq h - g$  on  $I$  and for all  $m, n$ . Let  $\epsilon > 0$  be given.

The function  $\varphi = h - g$  is  $M_\alpha$ -integrable function on  $I$ , and so there exists a gauge  $\delta_\varphi$  on  $I$  such that if  $D$  is a  $\delta_\varphi$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\left| S(\varphi, D) - \int_I \varphi \right| < \epsilon.$$

Since it is nonnegative, the function  $\varphi$  is also McShane integrable on  $I$ , which implies that the function  $\Phi(x) = \int_a^x \varphi$  is absolutely continuous on  $I$ ; that is, there exists a number  $\eta > 0$  such that if  $\{[a_i, b_i] : i = 1, 2, \dots, m\}$  is a finite collection of closed intervals in  $[a, b]$  with  $\sum_{i=1}^m (b_i - a_i) < \eta$ , then

$$\sum_{i=1}^m |\Phi(b_i) - \Phi(a_i)| < \epsilon.$$

Furthermore, by Egorov's Theorem [1, Theorem 2.13], there exists an open set  $O \subset I$  such that  $\mu(O) < \eta$  and  $f_n$  converges uniformly to  $f$  on  $I \setminus O$ . Choose an integer  $N > 0$  such that

$$|f_n(x) - f_m(x)| < \epsilon$$

for all  $m, n \geq N$  and for all  $x \in I \setminus O$ .

Define a gauge  $\delta_1$  on  $I$  as follows:

$$\delta_1(x) = \begin{cases} \delta_\varphi(x) & \text{if } x \in I \setminus O \\ \min\{\delta_\varphi(x), \text{dist}(x, O^c)\} & \text{if } x \in O. \end{cases}$$

Consider a  $\delta_1$ -fine  $M_\alpha$ -partition  $D$  of  $I$ , and integers  $m, n \geq N$ . Let  $D_1$  be the subset of  $D$  that has tags in  $I \setminus O$ , and let  $D_2 = D \setminus D_1$ . Also, let  $I_1 = \bigcup\{[u, v] : ([u, v], \xi) \in D_1\}$  and  $I_2 = \bigcup\{[u, v] : ([u, v], \xi) \in D_2\}$ . Using Saks-Henstock Lemma [2, Lemma 2.5] and the fact that  $\mu(I_2) < \eta$ ,

we obtain

$$\begin{aligned} & |S(f_n, D) - S(f_m, D)| \\ & \leq |S(f_n, D_1) - S(f_m, D_1)| + |S(f_m, D_2) - S(f_m, D_2)| \\ & < \epsilon\mu(I_1) + S(\varphi, D_2) \\ & \leq \epsilon(b-a) + \left| S(\varphi, D_2) - \int_{I_2} \varphi \right| + \sum_{([u,v],\xi) \in D_2} |\Phi(v) - \Phi(u)| \\ & < \epsilon(b-a) + 2\epsilon \end{aligned}$$

and

$$\begin{aligned} \left| \int_I f_n - \int_I f_m \right| & \leq \left| \int_{I_1} f_n - S(f_n, D_1) \right| + \left| \int_{I_2} f_n - S(f_n, D_2) \right| \\ & \quad + \left| \int_{I_1} f_m - S(f_m, D_1) \right| + \left| \int_{I_2} f_m - S(f_m, D_2) \right| \\ & \quad + |S(f_n, D) - S(f_m, D)| \\ & < 6\epsilon + \epsilon(b-a). \end{aligned}$$

Since each  $f_n$  is  $M_\alpha$ -integrable on  $I$ , there exists a gauge  $\delta \leq \delta_1$  on  $I$  such that if  $D$  is any  $\delta$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\left| S(f_n, D) - \int_I f_n \right| < \epsilon$$

for  $1 \leq n \leq N$ , and

$$\begin{aligned} & \left| S(f_n, D) - \int_I f_n \right| \\ & \leq |S(f_n, D) - S(f_N, D)| + \left| S(f_N, D) - \int_I f_N \right| + \left| \int_I f_N - \int_I f_n \right| \\ & < 9\epsilon + 2\epsilon(b-a) \end{aligned}$$

for  $n > N$ . Therefore, the sequence  $\{f_n\}$  is equi-integrable on  $I$ , and the rest of the corollary follows from Corollary 2.2.  $\square$

**COROLLARY 2.4** (Monotone Convergence Theorem). *Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . If  $\{f_n\}$  is a monotone sequence and  $\lim_{n \rightarrow \infty} \int_I f_n < \infty$ , then  $\{f_n\}$  is equi-integrable on  $I$ . Consequently, the function  $f$  is  $M_\alpha$ -integrable on  $I$ , and*

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

*Proof.* Assume that  $\{f_n\}$  is nondecreasing. The proof for the case when  $\{f_n\}$  is non-increasing is similar. Then, for each  $n$ , the function  $f_n - f_1$  is nonnegative and  $M_\alpha$ -integrable on  $I$ . It follows that each function  $f_n - f_1$  is McShane integrable on  $I$ . Since  $(f_n - f_1) \nearrow (f - f_1)$  and  $\lim_{n \rightarrow \infty} \int_I (f_n - f_1) < \infty$ , by the Monotone Convergence Theorem for McShane Integral [1, Corollary 13.4], the function  $f - f_1$  is McShane integrable (and so  $M_\alpha$ -integrable) on  $I$ . Thus, the function  $f$  is  $M_\alpha$ -integrable on  $I$ . Since  $f_1(x) \leq f_n(x) \leq f(x)$  for all  $x \in I$  and for all  $n$ , applying Corollary 2.3 completes the proof.  $\square$

### 3. Proof of the main theorem

We will use the following lemma.

**LEMMA 3.1.** *Let  $\{f_n\}$  be a sequence of  $M_\alpha$ -integrable functions on  $I$  such that  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ . If  $\{f_n\}$  is  $M_\alpha$ -convergent in Gordon's or Bartle's sense, then  $\{\int_I f_n\}$  is a Cauchy sequence in  $\mathbb{R}$ .*

*Proof.* Suppose  $\{f_n\}$  is  $M_\alpha$ -convergent in Gordon's sense on  $I$ . Let  $\epsilon > 0$  be given. Then there exists a gauge  $\delta$  on  $I$ , and we can choose a particular  $\delta$ -fine  $M_\alpha$ -partition  $D_0$  of  $I$  such that

$$\left| S(f_n, D_0) - \int_I f_n \right| < \epsilon \quad \text{for all } n \geq N_{D_0}$$

for some positive integer  $N_{D_0}$ , as guaranteed in the definition of  $M_\alpha$ -convergent sequence in Gordon's sense. Since  $D_0$  is finite and  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ , there exists an integer  $N \geq N_{D_0}$  such that

$$|S(f_n, D_0) - S(f_m, D_0)| < \epsilon \quad \text{for all } m, n \geq N,$$

which implies that

$$\begin{aligned} & \left| \int_I f_n - \int_I f_m \right| \\ & \leq \left| \int_I f_n - S(f_n, D_0) \right| + |S(f_n, D_0) - S(f_m, D_0)| + \left| S(f_m, D_0) - \int_I f_m \right| \\ & < 3\epsilon. \end{aligned}$$

Thus,  $\{\int_I f_n\}$  is a Cauchy sequence in  $\mathbb{R}$ .

On the other hand, suppose  $\{f_n\}$  is  $M_\alpha$ -convergent to  $f$  in Bartle's sense on  $I$ . Let  $\epsilon > 0$  be given, and let  $N_\epsilon$  be the positive integer guaranteed by the definition of  $M_\alpha$ -convergence in Bartle's sense. If  $m$

and  $n$  are integers such that  $m, n \geq N_\epsilon$ , then there exist gauges  $\delta_m$  and  $\delta_n$  on  $I$  such that, for every  $\delta_m$ -fine  $M_\alpha$ -partition  $D_m$  of  $I$ , we have

$$|S(f_m, D_m) - S(f, D_m)| < \epsilon,$$

and, for every  $\delta_n$ -fine  $M_\alpha$ -partition  $D_n$  of  $I$ , we have

$$|S(f_n, D_n) - S(f, D_n)| < \epsilon.$$

Furthermore, since  $f_m$  and  $f_n$  are  $M_\alpha$ -integrable on  $I$ , there exist gauges  $\delta'_m, \delta'_n$  on  $I$  such that, for every  $\delta'_m$ -fine  $M_\alpha$ -partition  $D'_m$  of  $I$ ,

$$\left| S(f_m, D'_m) - \int_I f_m \right| < \epsilon$$

and, for every  $\delta'_n$ -fine  $M_\alpha$ -partition  $D'_n$  of  $I$ ,

$$\left| S(f_n, D'_n) - \int_I f_n \right| < \epsilon.$$

Set  $\delta_\epsilon(x) = \min\{\delta_m(x), \delta_n(x), \delta'_m(x), \delta'_n(x)\}$  for  $x \in I$ . Therefore, if  $D$  is  $\delta_\epsilon$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\begin{aligned} \left| \int_I f_n - \int_I f_m \right| &\leq \left| \int_I f_n - S(f_n, D) \right| + |S(f_n, D) - S(f, D)| \\ &\quad + |S(f, D) - S(f_m, D)| + \left| S(f_m, D) - \int_I f_m \right| \\ &< 4\epsilon, \end{aligned}$$

and so  $\{\int_I f_n\}$  is a Cauchy sequence in  $\mathbb{R}$ . □

We are now ready to prove the Main Theorem (Theorem 2.1). We break the proof into four parts.

**(i)  $\implies$  (ii).** Suppose that  $f$  is  $M_\alpha$ -integrable on  $I$ , and

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

Let  $\epsilon > 0$  be given. Then there is an integer  $N > 0$  such that

$$\left| \int_I f - \int_I f_n \right| < \frac{\epsilon}{3} \quad \text{for all } n \geq N,$$

and there exists a gauge  $\delta$  on  $I$  such that if  $D$  is a  $\delta$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\left| S(f, D) - \int_I f \right| < \frac{\epsilon}{3}.$$



Since each such  $D$  is finite and since  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ , we can choose for each  $D$  an integer  $N_D \geq N$  such that

$$|S(f_n, D) - S(f, D)| < \frac{\epsilon}{3} \quad \text{for all } n \geq N_D.$$

Thus, for every integer  $n \geq N_D$ , we have

$$\begin{aligned} & \left| S(f_n, D) - \int_I f_n \right| \\ & \leq |S(f_n, D) - S(f, D)| + \left| S(f, D) - \int_I f \right| + \left| \int_I f - \int_I f_n \right| \\ & < \epsilon, \end{aligned}$$

and so  $\{f_n\}$  is  $M_\alpha$ -convergent in Gordon's sense on  $I$ .

(ii)  $\implies$  (i). Suppose that  $\{f_n\}$  is  $M_\alpha$ -convergent in Gordon's sense on  $I$ . By Lemma 3.1, the sequence  $\{\int_I f_n\}$  is Cauchy in  $\mathbb{R}$ , and so it converges to some real number  $A$ . Let  $\epsilon > 0$  be given. Then there exists an integer  $N > 0$  such that

$$\left| \int_I f_n - A \right| < \frac{\epsilon}{3} \quad \text{for all } n \geq N,$$

and, as guaranteed in the definition of  $M_\alpha$ -convergence in Gordon's sense, there exists a gauge  $\delta$  on  $I$  such that if  $D$  is a  $\delta$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\left| \int_I f_n - S(f_n, D) \right| < \frac{\epsilon}{3} \quad \text{for all } n \geq N_D$$

for some integer  $N_D \geq N$ . Since each such  $D$  is finite and since  $f_n(x) \rightarrow f(x)$  for all  $x \in I$ , there is an integer  $n_1 \geq N_D$  such that

$$|S(f, D) - S(f_{n_1}, D)| < \frac{\epsilon}{3}.$$

Therefore, if  $D$  is any  $\delta$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\begin{aligned} & |S(f, D) - A| \\ & \leq |S(f, D) - S(f_{n_1}, D)| + \left| S(f_{n_1}, D) - \int_I f_{n_1} \right| + \left| \int_I f_{n_1} - A \right| \\ & < \epsilon, \end{aligned}$$

which implies that  $f$  is  $M_\alpha$ -integrable to  $A$  on  $I$ , and

$$\int_I f = A = \lim_{n \rightarrow \infty} \int_I f_n.$$

**(i)  $\implies$  (iii).** Suppose that  $f$  is  $M_\alpha$ -integrable on  $I$ , and

$$\int_I f = \lim_{n \rightarrow \infty} \int_I f_n.$$

Let  $\epsilon > 0$  be given. Then there is an integer  $N > 0$  such that

$$\left| \int_I f - \int_I f_n \right| < \frac{\epsilon}{3} \quad \text{for all } n \geq N.$$

Since  $f_n$  is  $M_\alpha$ -integrable on  $I$ , there is a gauge  $\gamma_n$  on  $I$  such that, for every  $\gamma_n$ -fine  $M_\alpha$ -partition  $D_n$  of  $I$ ,

$$\left| S(f_n, D_n) - \int_I f_n \right| < \frac{\epsilon}{3}.$$

Since  $f$  is  $M_\alpha$ -integrable on  $I$ , there is a gauge  $\delta$  on  $I$  such that, for every  $\delta$ -fine  $M_\alpha$ -partition  $D$  of  $I$ ,

$$\left| S(f, D) - \int_I f \right| < \frac{\epsilon}{3}.$$

Let  $\delta_n = \min\{\delta, \gamma_n\}$ . For  $n \geq N$ , if  $D$  is  $\delta_n$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\begin{aligned} |S(f_n, D) - S(f, D)| &\leq \left| S(f_n, D) - \int_I f_n \right| + \left| \int_I f_n - \int_I f \right| \\ &\quad + \left| \int_I f - S(f, D) \right| \\ &< \epsilon, \end{aligned}$$

which implies that  $\{f_n\}$  is  $M_\alpha$ -convergent to  $f$  in Bartle's sense on  $I$ .

**(iii)  $\implies$  (i).** Suppose that  $\{f_n\}$  is  $M_\alpha$ -convergent to  $f$  in Bartle's sense on  $I$ . By Lemma 3.1, the sequence  $\{\int_I f_n\}$  is Cauchy in  $\mathbb{R}$ , and so it converges to some real number  $A$ . Let  $\epsilon > 0$  be given. Then there exists an integer  $N_\epsilon > 0$  such that

$$\left| \int_I f_{N_\epsilon} - A \right| < \frac{\epsilon}{3},$$

and there is a gauge  $\delta_{N_\epsilon}$  on  $I$  such that, for every  $\delta_{N_\epsilon}$ -fine  $M_\alpha$ -partition  $D$  of  $I$ , we have

$$|S(f, D) - S(f_{N_\epsilon}, D)| < \frac{\epsilon}{3}.$$

Moreover, since  $f_{N_\epsilon}$  is  $M_\alpha$ -integrable on  $I$ , there is a gauge  $\gamma_{N_\epsilon}$  on  $I$  such that, for every  $\gamma_{N_\epsilon}$ -fine  $M_\alpha$ -partition  $D$  of  $I$ ,

$$\left| S(f_{N_\epsilon}, D) - \int_I f_{N_\epsilon} \right| < \frac{\epsilon}{3}.$$

Let  $\delta_\epsilon(\cdot) = \min\{\delta_{N_\epsilon}(\cdot), \gamma_{N_\epsilon}(\cdot)\}$ . If  $D$  is  $\delta_\epsilon$ -fine  $M_\alpha$ -partition of  $I$ , then

$$\begin{aligned} & |S(f, D) - A| \\ & \leq |S(f, D) - S(f_{N_\epsilon}, D)| + \left| S(f_{N_\epsilon}, D) - \int_I f_{N_\epsilon} \right| + \left| \int_I f_{N_\epsilon} - A \right| \\ & < \epsilon, \end{aligned}$$

which implies that  $f$  is  $M_\alpha$ -integrable to  $A$  on  $I$ , and

$$\int_I f = A = \lim_{n \rightarrow \infty} \int_I f_n.$$

This ends the proof of the theorem. □

#### 4. An example

To exhibit the importance of the main theorem, we now give a sequence of  $M_\alpha$ -integrable functions that is not equi-integrable but is  $M_\alpha$ -convergent to a function in Bartle’s sense.

For each positive integer  $n$ , define

$$f_n(x) = \begin{cases} n & \text{if } x \in (1/n, 2/n), \\ -n & \text{if } x \in (2/n, 3/n), \\ 0 & \text{if } x \in [0, 3] \setminus \{(1/n, 2/n) \cup (2/n, 3/n)\}. \end{cases}$$

It is not difficult to compute that, for each  $n$ ,  $f_n$  is  $M_\alpha$ -integrable to 0 on  $[0, 3]$ .

CLAIM 1.  $\{f_n\}$  is  $M_\alpha$ -convergent to 0 in Bartle’s sense.

To see this, it is not difficult to observe that

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \text{for each } x \in [0, 3].$$

Let  $\epsilon > 0$ , and choose  $N_\epsilon$  to be a positive integer such that  $4N_\epsilon^{-1} < \epsilon$ . For every positive integer  $n \geq N_\epsilon$ , we define a gauge on  $I$  as follows:

$$\delta_n(x) = \begin{cases} \frac{1}{2} \text{dist}(x, \{1/n, 2/n, 3/n\}) & \text{if } x \notin \{1/n, 2/n, 3/n\}, \\ n^{-2} & \text{otherwise.} \end{cases}$$

If  $D = \{(I, t)\}$  be  $\delta_n$ -fine  $M_\alpha$ -partition of  $I$ , we have

$$|S(f_n, D)| \leq \frac{4}{n} \leq \frac{4}{N_\epsilon} < \epsilon,$$

which ends the proof of Claim 1.

CLAIM 2. Let  $0 < \epsilon < 1$ . For any gauge  $\delta$  on  $[0, 3]$ , there is a  $\delta$ -fine partial  $M_\alpha$ -partition  $\hat{D} = \{([u, v], \xi)\}$  of  $[0, 3]$  such that, for some positive integer  $n$ ,

$$\left| S(f_n, \hat{D}) - \int_{\cup [u,v]} f_n \right| > \epsilon.$$

To prove this claim, let  $\delta$  be an arbitrary gauge on  $[0, 3]$ , and choose a positive integer  $n$  such that  $2n^{-1} < \delta(0)$ . Then  $\hat{D} = \{([0, 2n^{-1}], 0)\}$  is  $\delta$ -fine partial  $M_\alpha$ -partition of  $[0, 3]$ , and

$$\left| S(f_n, \hat{D}) - \int_{[0, 2n^{-1}]} f_n \right| = |f_n(0)(2/n) - 1| = |0 - 1| = 1 > \epsilon,$$

which ends the proof of the second claim.

CLAIM 3.  $\{f_n\}$  is not equi-integrable.

To prove this, we suppose that  $\{f_n\}$  is equi-integrable. Let  $0 < \epsilon < 1$  be given. Then there is a gauge  $\delta$  on  $[0, 3]$  such that for any  $\delta$ -fine  $M_\alpha$ -partition  $D$  of  $[0, 3]$

$$\left| S(f_n, D) - \int_{[0,3]} f_n \right| < \epsilon \text{ for all positive integer } n.$$

By Saks-Henstock Lemma [2, Lemma 2.5], for any  $\delta$ -fine partial  $M_\alpha$ -partition  $\hat{D} = \{(I_i, t_i)\}$  of  $[0, 3]$ , we get

$$\left| S(f_n, \hat{D}) - \int_{\cup I_i} f_n \right| < \epsilon \text{ for all positive integer } n,$$

a contradiction to Claim 2. □

### References

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