2020

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To cite this article: B C L Felipe et al 2020 J. Phys.: Conf. Ser. 1538 012009

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On the set chromatic number of the join and comb product of graphs

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Abstract. A vertex coloring $c : V(G) \to \mathbb{N}$ of a non-trivial connected graph $G$ is called a set coloring if $NC(u) \neq NC(v)$ for any pair of adjacent vertices $u$ and $v$. Here, $NC(x)$ denotes the set of colors assigned to vertices adjacent to $x$. The set chromatic number of $G$, denoted by $\chi_s(G)$, is defined as the fewest number of colors needed to construct a set coloring of $G$. In this paper, we study the set chromatic number in relation to two graph operations: join and comb product. We determine the set chromatic number of wheels and the join of a bipartite graph and a cycle, the join of two cycles, the join of a complete graph and a bipartite graph, and the join of two bipartite graphs. Moreover, we determine the set chromatic number of the comb product of a complete graph with paths, cycles, and large star graphs.

1. Introduction
A vertex or edge coloring $c$ of a graph $G$ is said to be neighbor-distinguishing if $c$ induces a vertex labelling in which every pair of adjacent vertices in $G$ is assigned distinct labels. Aside from the well-studied proper vertex coloring, various neighbor-distinguishing colorings have also been introduced and discussed in the literature (for examples, see [3], [4], [6], [7], [8]).

In [5], Chartrand, Okamoto, and Zhang introduced and explored the following neighbor-distinguishing vertex coloring that makes use of neighborhood color sets.

Definition 1.1. [5] For a nontrivial connected graph $G$, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of $G$ where adjacent vertices may be assigned the same color.

(i) For a set $S \subseteq V(G)$, define the set $c(S)$ of colors assigned to the vertices of $S$ by

$$c(S) = \{c(v) : v \in S\}.$$ 

(ii) The neighborhood color set of $v$, denoted by $NC(v)$, is the set $c(N(v))$; that is, $NC(v)$ is the set of colors of the neighbors of $v$.

(iii) The coloring $c$ is called set neighbor-distinguishing, or simply a set coloring, if $NC(u) \neq NC(v)$ for every pair $u, v$ of adjacent vertices of $G$.

(iv) The minimum number of colors required in a set coloring of $G$ is called the set chromatic number of $G$ and is denoted by $\chi_s(G)$.

In their paper, Chartrand et al. studied the set chromatic numbers of some families of graphs and established some bounds for the set chromatic number in terms of other graph parameters.
It is clear that $\chi_s(G) \leq \chi(G)$. Moreover, results from [5] on the lower bound for $\chi_s$ are presented below.

**Proposition 1.2.** [5] For every graph $G$,

(i) $\chi_s(G) \geq \lceil \log_2(\chi(G) + 1) \rceil$,

(ii) $\chi_s(G) \geq 1 + \lceil \log_2(\omega(G)) \rceil$,

where $\chi(G)$ and $\omega(G)$ are the chromatic and clique numbers of $G$, respectively.

In this paper, we will study the set chromatic number of graphs in relation to two well-studied graph operations: join and comb product. Section 2 will focus on the join of graphs. Our results cover wheels and the join of a bipartite graph and a cycle, the join of two cycles, the join of a complete graph and a bipartite graph, and the join of two bipartite graphs. Meanwhile, section 3 focuses on the comb product of graphs. We will determine the set chromatic number of the comb product of a complete graph with paths, cycles, and large star graphs.

2. On the set chromatic number of the join of graphs

We present the definition of the join of two graphs below:

**Definition 2.1.** Let $G$ and $H$ be two vertex-disjoint graphs. The join $G + H$ of $G$ and $H$ is the graph whose vertex and edge sets are given by

$V(G + H) = V(G) \cup V(H)$,

$E(G + H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$.

It is well-known that $\chi(G + H) = \chi(G) + \chi(H)$. In relation to the set colorings, the join of graphs has been studied in [10] by Okamoto et al. In particular, they have established the following sharp bounds:

**Theorem 2.2.** [10] For every two graphs $G$ and $H$,

$$\max\{\chi_s(G) + \lceil \log_2 \omega(H) \rceil, \chi_s(H) + \lceil \log_2 \omega(G) \rceil\} \leq \chi_s(G + H) \leq \chi_s(G) + \chi_s(H) + 1,$$

where $\omega(\cdot)$ denotes the clique number of a graph.

Moreover, they also considered the join of graphs with complete graphs.

**Theorem 2.3.** [10] For a graph $G$ and a positive integer $p$,

$$\chi_s(G) + p - 1 \leq \chi_s(G + K_p) \leq \chi_s(G) + p.$$

While Theorems 2.2 and 2.3 focused on bounds for the set chromatic number of joins, our results focus on the exact set chromatic number of a join of two graphs from well-known graph families. We begin with the following.

**Proposition 2.4.** Let $W_n$ be the wheel graph of order $n$. Then $\chi_s(W_4) = 4$ and $\chi_s(W_n) = 3$ for $n \geq 5$.

**Proof.** The wheel graph $W_4$ is a complete graph of order 4. Hence, $\chi_s(W_4) = 4$. For $n \geq 5$, without loss of generality, suppose $W_n = C_n + K_1$ such that $C_n = v_1, v_2, \ldots, v_n, v_1$ and $V(K_1) = \{v_0\}$. Suppose $n$ is even; then $\chi(W_n) = 3$, which implies that $\chi_s(W_n) = 3$ as well. Now, suppose $n$ is odd. Then $\chi(W_n) = 4$ and $\chi_s(W_n) \geq 3$. We define a coloring $c : V(W_n) \to \{1, 2, 3\}$ as follows:

$$c(v_i) = \begin{cases} 1, & i \in \{0, 1, 2, n\}, \\ 2, & i \text{ is odd and } 3 \leq i \leq n - 1, \\ 3, & i \text{ is even and } 3 \leq i \leq n - 1. \end{cases}$$

Then (refer to Fig. 1), it is easy to see that $c$ is a set 3-coloring of $W_n$. Therefore, $\chi_s(W_n) = 3$ for $n \geq 5$. $\square$
Figure 1. A set 3-coloring of $W_n$, with odd $n \geq 5$.

We now consider the join of a bipartite graph with another bipartite graph, an odd cycle, or a complete graph.

**Theorem 2.5.** Let $B$ be a bipartite graph.

(i) If $H$ is also a bipartite graph, then $\chi_s(B + H) = 4$.

(ii) If $C_n$ is an odd cycle, then $\chi_s(B + C_3) = 5$ and $\chi_s(B + C_n) = 4$ for $n \geq 5$.

(iii) If $p$ is a positive integer, then $\chi_s(B + K_p) = p + 2$.

**Proof.** (i) Since $\chi(B + H) = 4$, it follows that $\chi_s(B + H)$ is 3 or 4. Suppose there is a set 3-coloring $c$ of $B + H$; then $c_1 := c|_{V(B)}$ and $c_2 := c|_{V(H)}$ are set colorings of $B$ and $H$, respectively. Clearly, neither $c_1$ nor $c_2$ can use all 3 colors; so we can assume that $c_1$ uses colors 1 and 2 while $c_2$ uses colors 1 and 3.

As shown in Fig. 2, if $v \in V(B)$, then $\text{NC}(v) \in \{\{1, 3\}, \{1, 2, 3\}\}$. On the other hand, if $w \in V(H)$, then $\text{NC}(w) \in \{\{1, 2\}, \{1, 2, 3\}\}$. Since $\chi(B) = \chi(H) = 2$, there must be a vertex $p$ in $B$ and a vertex $q$ in $H$ whose $\text{NC}$ is $\{1, 2\}$. Since $p$ and $q$ are adjacent in $B + H$, this proves that we cannot use 3 colors. Therefore, $\chi_s(B + H) = 4$.

(ii) First, we have

$$4 = \chi_s(C_3) + \lceil \log_2 \omega(B) \rceil \leq \chi_s(B + C_3) \leq \chi(B + C_3) = 5.$$ 

Then $\chi_s(B + C_3) = 4$ or 5. By using an argument similar to that in (i), we can show that $B + C_3$ has no set 4-coloring; hence, $\chi_s(B + C_3) = 5$.

Now, consider $B + C_n$, where $n \geq 4$. Then $\chi_s(B + C_n) \geq \chi_s(C_n) + \lceil \log_2 \omega(B) \rceil = 4$.

Suppose $B$ has partite sets $V_1$ and $V_2$ and $C_n = v_1, v_2, ..., v_n, v_1$. Define a coloring $c : V(G) \to \{1, 2, 3, 4\}$ as follows:
By Theorem 2.3, $p$ is an odd positive integer. Then $\chi_s(B + C_n) = 4$.

Then, it is easy to see that $c$ is a set 4-coloring of $B + C_n$; refer to Fig. 3. Hence, $\chi_s(B + C_n) = 4$.

Figure 3. A set 4-coloring of $B + C_n$, $n \geq 4$.

(iii) By Theorem 2.3, $p + 1 \leq \chi_s(B + K_p) \leq p + 2$. Suppose $c$ is a set coloring of $B + K_p$ that uses only $p + 1$ colors. Then $|c(K_p)| = p$ and we can assume that $c(K_p) = N_p := \{1, 2, \ldots, p\}$.

Let $B_1$ and $B_2$ be the partite sets of $B$. Then there is a non-isolated vertex $v$ in $B_1$ (without loss of generality) for which $c(v) = p + 1$. If $y \in B_2$ is a neighbor of $v$, then $c(y) \in N_p$. Suppose $c(y) = j \in N_p$. Let $z$ be the vertex in $K_p$ whose color is also $j$. Then $NC(z) = N_{p+1} = NC(y)$, which is not possible. Therefore, $\chi_s(B + K_p) = p + 2$.

We also determine the set chromatic number of the join of two odd cycles.

**Proposition 2.6.** Let $n, k \geq 3$ be odd positive integers. Then

$$\chi_s(C_n + C_k) = \begin{cases} 6, & \text{if } n = k = 3, \\ 5, & \text{otherwise.} \end{cases}$$

**Proof.** Suppose $C_n = v_1, v_2, \ldots, v_n, v_1$ and $C_k = u_1, u_2, \ldots, u_k, u_1$. Let $G = C_n + C_k$. Then $\chi_s(G) \geq \chi_s(C_n) + \lceil \log_2 \omega(C_k) \rceil = 4$. It is easy to show that $G$ is not set 4-colorable. We now consider four cases:

Case 1: $n = k = 3$. Since $G = C_3 + C_3 \cong K_6$, $\chi_s(G) = 6$.

Case 2: $n = 3$ and $k \geq 5$. Define a coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ as follows:

$$c(v_i) = \begin{cases} 1, & i = 1, \\ 2, & i = 2, \\ 3, & i = 3, \end{cases} \quad \text{and} \quad c(u_i) = \begin{cases} 1, & i \in \{1, 2, k\}, \\ 4, & i \text{ is odd and } 3 \leq i \leq k - 1, \\ 5, & i \text{ is even and } 3 \leq i \leq k - 1. \end{cases}$$

Then (refer to Fig. 4), it is easy to see that $c$ is a set 5-coloring of $G$. Therefore, $\chi_s(C_3 + C_k) = 5$ for $k \geq 5$.

Case 3: $n, k \geq 5$. Define a coloring $f : V(G) \to \{1, 2, 3, 4, 5\}$ as follows:

$$f(v_i) = \begin{cases} 1, & i \in \{1, 2, n\}, \\ 2, & i \text{ is odd, } 3 \leq i \leq n - 1, \quad \text{and} \quad f(u_i) = \begin{cases} 1, & i \in \{1, 2, k\}, \\ 4, & i \text{ is odd, } 3 \leq i \leq k - 1, \\ 5, & i \text{ is even, } 3 \leq i \leq k - 1. \end{cases} \\ 3, & i \text{ is even, } 3 \leq i \leq k - 1, \end{cases}$$

Then (refer to Fig. 4), it is easy to see that $c$ is a set 5-coloring of $G$. Therefore, $\chi_s(C_3 + C_k) = 5$ for $k \geq 5$.

Case 4: $n, k \geq 5$. Define a coloring $f : V(G) \to \{1, 2, 3, 4, 5\}$ as follows:

$$f(v_i) = \begin{cases} 1, & i \in \{1, 2, n\}, \\ 2, & i \text{ is odd, } 3 \leq i \leq n - 1, \quad \text{and} \quad f(u_i) = \begin{cases} 1, & i \in \{1, 2, k\}, \\ 4, & i \text{ is odd, } 3 \leq i \leq k - 1, \\ 5, & i \text{ is even, } 3 \leq i \leq k - 1. \end{cases} \\ 3, & i \text{ is even, } 3 \leq i \leq k - 1, \end{cases}$$

Then (refer to Fig. 4), it is easy to see that $c$ is a set 5-coloring of $G$. Therefore, $\chi_s(C_3 + C_k) = 5$ for $k \geq 5$. 
Then (refer to Fig. 5), it is easy to see that \( c \) is a set 5-coloring of \( G \). Therefore, \( \chi_s(C_n + C_k) = 5 \) for \( n, k \geq 5 \).
**Corollary 3.3.** Let \( n \geq 2, m \geq 2 \), and \( r \) be a pendant vertex of \( P_m \). Then \( \chi_s(K_n \triangleright_r P_m) = n \).

The comb product of graphs is a well-studied graph operation and has been considered in relation to several graph parameters and labellings (for examples, see [1], [2], [9], [11], [12]).

In this work, we focus on the comb product of complete graphs with well-known graph families. We present below our first result, which concerns the comb products of complete graphs with cycles or paths. The proof is based on a technique used for the proof of Proposition 3.1 in [5].

**Theorem 3.4.** Let \( H \) be a cycle \( C_s \) (\( s \geq 4 \)) rooted at any vertex \( r \), or a path \( P_t \) (\( t \geq 3 \)) rooted at a vertex \( r \) that is not a leaf. Then for \( n \geq 3 \), \( \chi_s(K_n \triangleright_r H) = \left\lceil \frac{1}{2}(1 + \sqrt{8n + 1}) \right\rceil \).

**Proof.** Denote the quantity \( \left\lceil \frac{1}{2}(1 + \sqrt{8n + 1}) \right\rceil \) by \( k_n \). First, notice that in constructing \( K_n \triangleright_r H \), each vertex of \( K_n \) is connected to two nonadjacent vertices of a copy of \( H \). Thus, since \( \chi_s(H) \leq 3 \) and \( k_n \geq 3 \), the result follows directly from the case where \( H = P_3 \), rooted at a non-leaf vertex \( r \).

Now, suppose \( H = P_3 \), rooted at a non-leaf vertex \( r \). For simplicity, we denote \( K_n \triangleright_r H \) by \( G_n \) and we label its vertices as shown in Figure 6.

![Figure 6. Labels of vertices in \( K_n \triangleright_r P_3 \).](image)

First, we show that \( \chi_s(G_n) \leq k_n \) by constructing a set coloring \( c \) of \( G_n \) that uses at most \( k_n \) colors. Let \( S_1, S_2, ..., S_m \) be all the 1-subsets and 2-subsets of \( \{2, 3, ..., k_n\} \). Then \( m = \binom{k_n - 1}{1} + \binom{k_n - 1}{2} \) and it follows that \( m \geq n \). Hence, we can define the coloring \( c \) so that \( c(x) = 1 \) for all \( x = x_i, i = 1, 2, ..., n \), and \( \{c(x_{i,j}) : j = 1, 2\} = S_i \) for \( i = 1, 2, ..., n \). Thus, \( NC(x_i) = \{1\} \cup S_i \) and \( NC(x_{i,j}) = \{1\} \) for \( i = 1, 2, ..., n \) and \( j = 1, 2 \). Since the \( S_i \)'s are nonempty and disjoint, it follows that \( c \) is a set coloring.

We now show that \( \chi_s(G_n) \geq k_n \) by showing that any set coloring of \( G_n \) uses at least \( k_n \) colors. To this end, suppose \( c \) is a set coloring of \( G_n \) that uses \( p \) colors. We split the proof into three cases:

**Case 1.** Suppose \( c \) colors the vertices \( x_1, x_2, ..., x_n \) using \( n \) colors. Then \( p \geq n \). Since \( n \geq 3 \) implies \( n \geq k_n \), we must have \( p \geq k_n \).
Case 2. Suppose \( c(x) = 1 \) for all \( x = x_i, i = 1, 2, \ldots, n \). Then for each \( i \),

\[
NC(x_i) = \begin{cases} 
\{1\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1\}, \\
\{1, a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1, a\} \text{ or } \{a\}, \\
\{1, a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{a, b\},
\end{cases}
\]
where \( 1 < a, b \leq p \). Clearly, the case where \( NC(x_i) = \{1\} \) is not possible since this would also make \( NC(x_{i,j}) = \{1\} \) for \( j = 1, 2 \). Hence, since \( c \) is a set coloring, we should have at least \( n \) different neighborhood color sets of the form \( \{1, a\} \) or \( \{1, a, b\} \), where \( 1 < a, b \leq p \). Then \((p^{-1}) + (p^{-1}) \geq n\), which is equivalent to \( p \geq k_n \).

Case 3. Suppose \( c(x_1, x_2, \ldots, x_n) = N_\ell := \{1, \ldots, \ell\} \), where \( 2 \leq \ell \leq n-1 \). Let \( S \) be the subset of \( \{x_1, x_2, \ldots, x_n\} \) satisfying the following condition: for any \( x \in S \), there is a vertex \( y \neq x \) in \( S \) for which \( c(x) = c(y) \). For convenience, we assume that \( S = \{x_1, x_2, \ldots, x_{|S|}\} \). Since \( 2 \leq \ell \leq n-1 \), we must have \( |S| \geq n - \ell + 1 \). Moreover, for \( x_i \in S \),

\[
NC(x_i) = \begin{cases} 
N_\ell, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} \subseteq N_\ell, \\
N_\ell \cup \{a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - N_\ell = \{a\}, \\
N_\ell \cup \{a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - N_\ell = \{a, b\},
\end{cases}
\]
where \( \ell < a, b \leq p \). Since \( c \) is a set coloring, only at most one \( x_i \) in \( S \) can have \( NC(x_i) = N_\ell \). Hence, we should have at least \( |S| - 1 \) different neighborhood color sets of the form \( N_\ell \cup \{a\} \) or \( N_\ell \cup \{a, b\} \). Then \((p^{-\ell}) + (p^{-\ell}) \geq |S| - 1 \geq (n - \ell + 1) - 1 = n - \ell \), which is equivalent to

\[
p \geq \left[ \frac{2^{\ell-1}}{2} + \sqrt{2n + \frac{1}{4} - 2\ell} \right].
\]

Then \( p \geq \min_{\ell=2,3,\ldots,n-1} \left[ \frac{2\ell-1}{2} + \sqrt{2n + \frac{1}{4} - 2\ell} \right]. \) It can easily be verified that this minimum occurs when \( \ell = 2 \); hence

\[
p \geq \left[ \frac{3}{2} + \sqrt{2n - \frac{15}{4}} \right] \geq k_n,
\]

since \( n \geq 3 \).

Therefore, in any case, any set coloring of \( G_n \) that uses \( p \) colors must have \( p \geq k_n \). \( \square \)

In Theorem 3.4, the case where \( H \) is the cycle \( C_3 \) was not considered. This is because, in this case, in constructing the comb product \( K_n \upharpoonright_r C_3 \), each vertex of \( K_n \) is connected to two adjacent vertices of a copy of \( C_3 \). For example, the comb product \( K_3 \upharpoonright_r C_3 \) has set chromatic number 3; an optimal set coloring is shown in Figure 7. We consider the rest of this exceptional case in Proposition 3.5.

**Proposition 3.5.** Let \( H \) be the cycle \( C_3 \), rooted at any vertex \( r \). Then for \( n \geq 4 \), \( \chi_s(K_n \upharpoonright_r H) = \left[ \frac{1}{2}(3 + \sqrt{8n - 7}) \right] \).

**Proof.** Denote the quantity \( \left[ \frac{1}{2}(3 + \sqrt{8n - 7}) \right] \) by \( k_n \) and the graph \( K_n \upharpoonright_r H \) by \( G_n \). First, note that \( k_n \geq 4 \) for \( n \geq 4 \). We show that \( \chi_s(G_n) \leq k_n \) by constructing a set coloring \( c \) of \( G_n \) that uses at most \( k_n \) colors. Let \( S_1, S_2, \ldots, S_m \) be all the 1-subsets and 2-subsets of \( \{3, 4, \ldots, k_n\} \). Then \( m = \binom{k_n - 2}{1} + \binom{k_n - 2}{2} \) and since \( n \geq 4 \), it follows that \( m \geq n - 1 \). Hence, we can define the coloring \( c \) so that \( c(x_1) = 2, c(x) = 1 \) for all \( x = x_i (i = 2, 3, \ldots, n) \), \( \{c(x_{1,j}) : j = 1, 2\} = \{3, 4\} \), and

\[
\{c(x_{i,j}) : j = 1, 2\} = \begin{cases} 
\{1\} \cup S_i, & \text{if } |S_i| = 1, \\
S_i, & \text{if } |S_i| = 2,
\end{cases}
\]
Case 1. Suppose \( C_1 \) implies \( n \) where \( 1 < a, b \). Since in this case, we would have \( \ell \geq a, b \), which is equivalent to \( \ell = 1 \). Let \( S_i \) be the subset of \( \{1, a, b\} \) satisfying the following condition: for any \( x \in S_i \), there is a vertex \( y \neq x \) in \( S \) for which \( c(x) = c(y) \). For convenience, we assume that \( S = \{x_1, x_2, ..., x_n\} \). Since \( 2 \leq \ell \leq n - 1 \), we must have \( |S| \geq n - \ell + 1 \). Moreover, for \( x_i \in S \),

\[
NC(x_i) = \begin{cases} 
\mathbb{N}_\ell, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} \subseteq \mathbb{N}_\ell, \\
\mathbb{N}_\ell \cup \{a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a\}, \\
\mathbb{N}_\ell \cup \{a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a, b\}, 
\end{cases}
\]

where \( \ell < a, b \leq p \).

Figure 7. An optimal set coloring of \( K_3 \succ C_3 \).

for \( i = 2, ..., n \). Thus, \( NC(x_1) = \{1, 3, 4\} \), \( NC(x_i) = \{1, 2\} \cup S_i, i = 2, 3, ..., n \), \( NC(x_{1,j}) : j = 1, 2 \} = \{2, 3\}, \{2, 4\} \), and

\[
\{NC(x_{i,j}) : j = 1, 2\} = \begin{cases} 
\{\{1\}, \{1, a\}\}, & \text{if } S_i = \{a\}, \\
\{\{1, a\}, \{1, b\}\}, & \text{if } S_i = \{a, b\}, 
\end{cases}
\]

where \( a \) and \( b \) are distinct integers from \( 3, 4, ..., k_n \). Since the \( S_i \)'s are nonempty and disjoint, it follows that \( c \) is a set coloring.

We now show that \( \chi_s(G_n) \geq k_n \) by showing that any set coloring of \( G_n \) uses at least \( k_n \) colors. To this end, suppose \( c \) is a set coloring of \( G_n \) that uses \( p \) colors. We split the proof into four cases:

Case 1. Suppose \( c \) colors the vertices \( x_1, x_2, ..., x_n \) using \( n \) colors. Then \( p \geq n \). Since \( n \geq 4 \) implies \( n \geq k_n \), we must have \( p \geq k_n \).

Case 2. Suppose \( c(x) = 1 \) for all \( x = x_i, i = 1, 2, ..., n \). Then for each \( i \),

\[
NC(x_i) = \begin{cases} 
\{1\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1\}, \\
\{1, a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1, a\} \text{ or } \{a\}, \\
\{1, a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{a, b\}, 
\end{cases}
\]

where \( 1 < a, b \leq p \). Clearly, the case where \( NC(x_i) = \{1\} \) is not possible since this would also make \( NC(x_{i,j}) = \{1\} \) for \( j = 1, 2 \). Moreover, the case where \( NC(x_i) = \{1, a\} \) is also not possible since in this case, we would have \( c(x_i) = c(x_{i,j}) \) for some \( j \) or \( c(x_{i,1}) = c(x_{i,2}) \). Hence, we should have at least \( n \) different neighborhood color sets of the form \( \{1, a, b\} \), where \( 1 < a, b \leq p \). Then \( (p-1)^n \geq n \), which is equivalent to \( p \geq \frac{1}{3} (3 + 6n + 1) \geq k_n \).

Case 3. Suppose \( c(\{x_1, x_2, ..., x_n\}) = N_\ell := \{1, ..., \ell\} \), where \( 2 \leq \ell \leq n - 1 \). Let \( S \) be the subset of \( \{x_1, x_2, ..., x_n\} \) satisfying the following condition: for any \( x \in S \), there is a vertex \( y \neq x \) in \( S \) for which \( c(x) = c(y) \). For convenience, we assume that \( S = \{x_1, x_2, ..., x_{|S|}\} \). Since \( 2 \leq \ell \leq n - 1 \), we must have \( |S| \geq n - \ell + 1 \). Moreover, for \( x_i \in S \),

\[
NC(x_i) = \begin{cases} 
\mathbb{N}_\ell, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} \subseteq \mathbb{N}_\ell, \\
\mathbb{N}_\ell \cup \{a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a\}, \\
\mathbb{N}_\ell \cup \{a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a, b\}, 
\end{cases}
\]

where \( \ell < a, b \leq p \).
• **Case 3.1** Suppose $\ell = 2$. Then none of the $x_i$’s can have $NC(x_i) = N_2$. Then we must have $inom{p - 2}{1} + \binom{p - 2}{2} \geq |S| \geq n - 1$, which implies $p \geq k_n$.

• **Case 3.2** Suppose $3 \leq \ell n - 1$. Then only at most one of the $x_i$’s can have $NC(x_i) = N_\ell$. Then we must have $inom{p - \ell}{1} + \binom{p - \ell}{2} \geq |S| - 1 \geq n - \ell$, which implies

\[
p \geq \left\lceil \frac{2\ell - 1}{2} + \frac{1}{2} \sqrt{8n - 8\ell + 1} \right\rceil \geq \min_{\ell = 3, \ldots, n-1} \left\lceil \frac{2\ell - 1}{2} + \frac{1}{2} \sqrt{8n - 8\ell + 1} \right\rceil.
\]

It can be easily verified that the minimum occurs when $\ell = 3$; hence,

\[
p \geq \left\lceil 2.5 + 0.5\sqrt{8n - 23} \right\rceil \geq k_n \text{ since } n \geq 4.
\]

Therefore, in any case, any set coloring of $G_n$ that uses $p$ colors must have $p \geq k_n$. 

We now determine the set chromatic number of the comb product of complete graphs with star graphs of large order.

**Proposition 3.6.** Let $n \geq 3$ and $H$ be the star graph $K_{1,q}$, $q \geq n$, rooted at the vertex $r$ that is not a leaf. Then $\chi_s(K_n \triangleright_r H) = [1 + \log_2(n + 1)]$.

**Proof.** Denote the quantity $[1 + \log_2(n + 1)]$ by $k_n$ and the graph $K_n \triangleright_r H$ by $G_n$. Label the vertices of $K_n$ as $x_1, x_2, \ldots, x_n$ and for each $i$, label the leaf vertices adjacent to $x_i$ as $x_{i1}, x_{i2}, \ldots, x_{iq}$.

First, we show that $\chi_s(G_n) \leq k_n$ by constructing a set coloring $c$ of $G_n$ that uses at most $k_n$ colors. Let $S_1, S_2, \ldots, S_m$ be the set of all nonempty subsets of $\{2, 3, \ldots, k_n\}$. Then $m = 2^{k_n} - 1$ and it follows that $m \geq n$. Hence, we can define the coloring $c$ so that $c(x) = 1$ for all $x = x_i$, $i = 1, 2, \ldots, n$, and $\{c(x_{ij}) : j = 1, 2, \ldots, q\} = S_1$ for $i = 1, 2, \ldots, n$. Thus, $NC(x_i) = \{1\} \cup S_i$ and $NC(x_{ij}) = \{1\}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, q$. Since the $S_i$’s are nonempty and disjoint, it follows that $c$ is a set coloring.

We now show that $\chi_s(G_n) \geq k_n$ by showing that any set coloring of $G_n$ uses at least $k_n$ colors. To this end, suppose $c$ is a set coloring of $G_n$ that uses $p$ colors. We split the proof into three cases:

**Case 1.** Suppose $c$ colors the vertices $x_1, x_2, \ldots, x_n$ using $n$ colors. Then $p \geq n$. Since $n \geq 3$ implies $n \geq k_n$, we must have $p \geq k_n$.

**Case 2.** Suppose $c(x) = 1$ for all $x = x_i$, $i = 1, 2, \ldots, n$. Then for each $i$,

\[
NC(x_i) = \begin{cases} \{1\}, & \text{if } \{c(x_{ij}) : j = 1, 2, \ldots, q\} = \{1\}, \\ \{1\} \cup T, & \text{if } \{c(x_{ij}) : j = 1, 2, \ldots, q\} = \{1\} \cup T \text{ or } T, \end{cases}
\]

where $T$ is a nonempty subset of $\{2, \ldots, p\}$. Clearly, the case where $NC(x_i) = \{1\}$ is not possible since this would also make $NC(x_{ij}) = \{1\}$ for all $j$. Hence, since $c$ is a set coloring, we should have at least $n$ different sets $T$. Then $2^{p-1} - 1 \geq n$, which is equivalent to $p \geq k_n$.

**Case 3.** Suppose $c(x_1, x_2, \ldots, x_n) = N_\ell := \{1, \ldots, \ell\}$, where $2 \leq \ell \leq n-1$. Let $S$ be the subset of $\{x_1, x_2, \ldots, x_n\}$ satisfying the following condition: for any $x \in S$, there is a vertex $y \neq x$ in $S$ for which $c(x) = c(y)$. For convenience, we assume that $S = \{x_1, x_2, \ldots, x_{\ell}\}$. Since $2 \leq \ell \leq n-1$, we must have $|S| \geq n - \ell + 1$. Moreover, for $x_i \in S$,

\[
NC(x_i) = \begin{cases} N_\ell, & \text{if } \{c(x_{ij}) : j = 1, 2, \ldots, q\} \subseteq N_\ell, \\ N_\ell \cup T, & \text{if } \{c(x_{ij}) : j = 1, 2, \ldots, q\} - N_\ell = T, \end{cases}
\]
where $T$ is a nonempty subset of $\{\ell + 1, \ldots, p\}$. Since $c$ is a set coloring, only at most one $x_i$ in $S$ can have $NC(x_i) = N_\ell$. Hence, we should have at least $|S| - 1$ different sets $T$. Then $2^{\ell - 1} - 1 \geq |S| - 1 \geq (n - \ell + 1) - 1 = n - \ell$, which is equivalent to $p \geq \ell + \log_2(n - \ell + 1) \geq \min_{\ell=2,3,\ldots,n-1} (\ell + \log_2(n - \ell + 1))$. It can be easily verified that this minimum occurs when $\ell = 2$; hence

$$p \geq \lceil 2 + \log_2(n - 1) \rceil \geq k_n$$

since $n \geq 3$.

Therefore, in any case, any set coloring of $G_n$ that uses $p$ colors must have $p \geq k_n$.

4. Conclusion

In this paper, we considered the set chromatic number in relation to two well-studied graph operations: the join and the comb product. While previous research revealed sharp lower and upper bounds for the join of two graphs, our results pertained to the exact set chromatic numbers of the joins of two graphs from well-known families. Moreover, continuing a previous result on the corona of complete graphs, we determined the set chromatic number of the comb product of complete graphs with paths, cycles, and large star graphs.

Acknowledgment

The authors would like to thank the Office of the Vice President for the Loyola Schools and the Department of Mathematics BCA fund of Ateneo de Manila University for supporting our attendance to this conference. Moreover, we would like to thank the organizers of ICCGANT 2019 for warmly welcoming us and hosting a wonderful conference. Finally, we would like to thank the referees of this paper for their valuable comments and suggestions.

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