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The sigma chromatic number of the Sierpiński gasket graphs and the Hanoi graphs

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Abstract. A vertex coloring $c : V(G) \to \mathbb{N}$ of a non-trivial connected graph G is called a *sigma coloring* if $\sigma(u) \neq \sigma(v)$ for any pair of adjacent vertices u and v. Here, $\sigma(x)$ denotes the sum of the colors assigned to vertices adjacent to x. The *sigma chromatic number* of G, denoted by $\sigma(G)$, is defined as the fewest number of colors needed to construct a sigma coloring of G. In this paper, we determine the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs. Moreover, we prove the uniqueness of the sigma coloring for Sierpiński gasket graphs.

1. Introduction

In [4], Chartrand, Okamoto, and Zhang introduced a new kind of vertex coloring called a sigma coloring. It is defined as follows.

Definition 1.1 (Chartrand, Okamoto, Zhang [4]). *For a non-trivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G. For each $v \in V(G)$, the* ***color sum*** *of v, denoted by $\sigma(v)$, is defined to be the sum of the colors of the vertices adjacent to v. If $\sigma(u) \neq \sigma(v)$ for every two adjacent $u, v \in V(G)$, then c is called a* ***sigma coloring*** *of G. The minimum number of colors required in a sigma coloring of G is called its* ***sigma chromatic number*** *and is denoted by $\sigma(G)$.*

Sigma coloring is an example of a neighbor-distinguishing coloring, the most studied example of which is the proper vertex coloring. Over the years, various neighbor-distinguishing colorings have also been introduced and discussed in literature such as in [3] and [5]. The notion of sigma coloring is closely related to the vertex colorings/labellings, discussed in [1], [9], [12] that also use the sum of the colors/labels of a vertex's neighbors.

Sigma colorings of different families of graphs have already been studied. For instance, Chartrand et al. determined the sigma chromatic numbers of paths, cycles, bipartite, and complete multipartite graphs in [4]. More recently, in [10], Luzon, Ruiz, and Tolentino have determined the sigma chromatic numbers of some families of circulant graphs. The complexity of the sigma coloring problem has also been studied in [6].

In this paper, we determine the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs.

The **Sierpiński gasket graph** S_n, $n \geq 1$, is the graph whose vertices are the intersection points of the finite Sierpiński gasket and whose edges are the line segments of the gasket [13].

An alternative description of S_n can also be found in [8]. The construction of S_n is shown in Fig. 1 while a sigma coloring of S_2 is shown in Fig. 2.

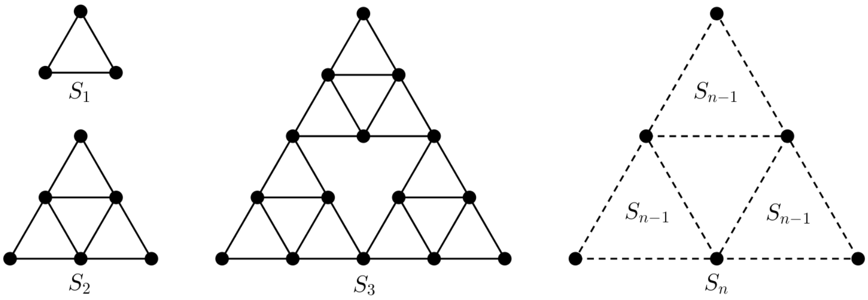

Figure 1. Sierpiński gasket graphs.

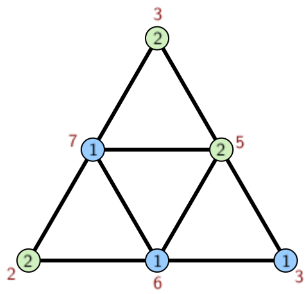

Figure 2. A sigma coloring of S_2.

On the other hand, the **Hanoi graphs** ([2], [7], [8]) are derived from the states of the Tower of Hanoi problem. Like the Sierpiński gasket graphs, the Hanoi graphs are also constructed in an iterative manner as shown in Fig. 3. A sigma 2-coloring of H_2 is shown in Fig. 4.

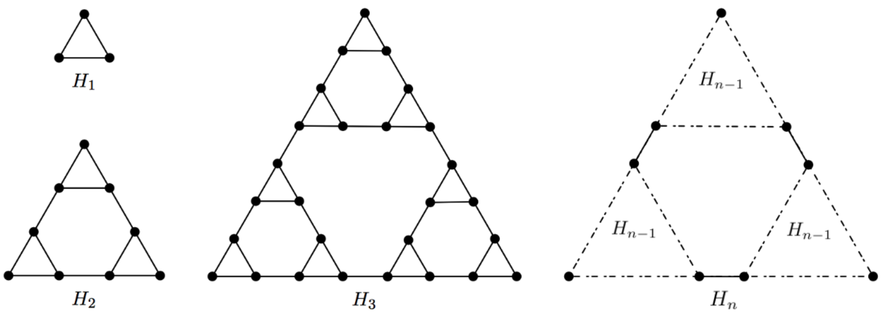

Figure 3. Hanoi graphs.

The (ordinary) chromatic numbers of S_n and H_n have already been determined by other researchers. For the Hanoi graphs, Parisse has constructed a natural coloring that leads to the following result.

Theorem 1.2 (Parisse, [11])**.** *For all $n \geq 1$, the chromatic number of H_n is 3.*

For the Sierpiński gasket graphs, Teguia and Godbole proved the following:

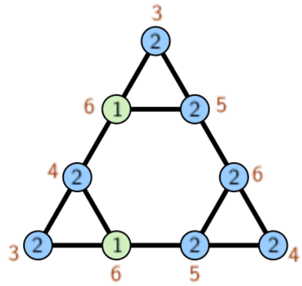

Figure 4. A sigma 2-coloring of H_2.

Theorem 1.3 (Teguia, Godbole [13])**.** *For all $n \geq 1$, the chromatic number of S_n is 3.*

Moreover, Klavžar proved the following stronger result.

Theorem 1.4 (Klavžar [8])**.** *The proper 3-coloring of S_n is unique for any $n \geq 1$.*

More precisely, Theorem 1.4 states that if c_1 and c_2 are two distinct proper 3-colorings of S_n, then it is possible to transform c_1 to c_2 by performing rotations, reflections, and/or a change of colors used.

In [4], Chartrand et al. also showed the following:

Theorem 1.5 (Chartrand, Okamoto, Zhang [4])**.** *For every graph G,*

$$\sigma(G) \leq \chi(G).$$

Hence, the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs are at most three. In this paper, we determine the exact sigma chromatic number of these graphs. Moreover, we also prove the uniqueness, in the same sense as in Theorem 1.4, of the sigma coloring of S_n ($n \geq 2$) that uses the minimum number of colors. The following are our main results.

Theorem 1.6.
(1) *For all $n \geq 3$, the sigma chromatic number of H_n is 3.*
(2) *For all $n \geq 2$, the sigma chromatic number of S_n is 2.*
(3) *For all $n \geq 2$, the sigma 2-coloring of S_n is unique.*

In Section 2, we prove statement (1) by showing that H_n, $n \geq 3$, does not have a sigma 2-coloring. In Section 3, we prove statement (2) by constructing iteratively a sigma 2-coloring of S_n, $n \geq 2$. Finally, in Section 4, we prove the uniqueness of this sigma 2-coloring.

2. The Sigma Chromatic Number of the Hanoi Graphs

In this section, we prove that the sigma chromatic number of the Hanoi graph H_n is 3 for all $n \geq 3$. We begin with the following proposition.

Proposition 2.1. *Let $n \geq 3$. If H_n is not sigma 2-colorable, then neither is H_{n+1}.*

Proof. Suppose that H_{n+1} has a sigma 2-coloring c that uses the colors a and b. Moreover, assume that a and b have been chosen so that the sets $\{3a, 2a+b, 2b+a, 3b\}$ and $\{2a, a+b, 2b\}$ are disjoint.

Let S be one of the three H_n subgraphs of H_{n+1} and define the coloring c' to be the restriction of c to S. We want to show that c' is a sigma 2-coloring of S.

To this end, let x and y be the two corner vertices of S that originally have degree three as vertices of H_{n+1}. Note that c and c' induce the same color sums for all vertices in $S \setminus \{x, y\}$. Hence, to show that c' is a sigma coloring of S, we need to focus only on the color sums of x and y. Now, since x and y are vertices with degree two in S, their color sums with respect to c' are in the set $\{2a, a+b, 2b\}$. Hence, by the assumption, these color sums cannot be equal to the color sum of any of the neighbors of x or y since these neighbors have degree three. This completes the proof. □

Thus, to prove that $\sigma(H_n) = 3, n \geq 3$, it is sufficient to prove that H_3 is not sigma 2-colorable. We show this in the following lemma.

Lemma 2.2. *The sigma chromatic number of H_3 is 3.*

Proof. Suppose there is a sigma 2-coloring $c : V(H_3) \to \{a, b\}$ of H_3. We show that c produces a contradiction. Let us label the vertices of H_3 as shown in Fig. 5.

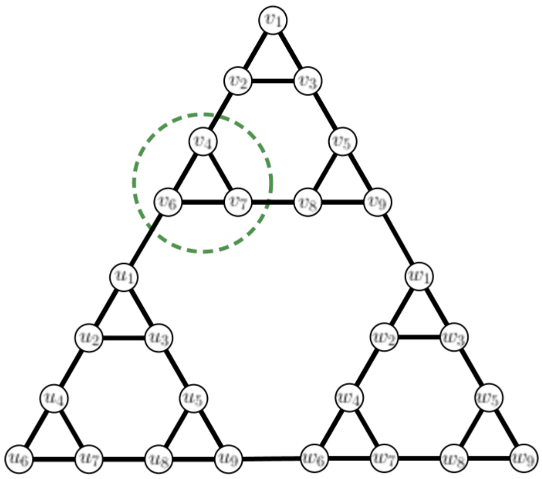

Figure 5. Vertex labels of the Hanoi graph H_3.

We consider cases based on the possible values of $c(v_4), c(v_6)$, and $c(v_7)$. There are four cases, in each of which we show how the contradiction arises.

Case 1 $\big(c(v_4), c(v_6), c(v_7)\big) = (a, a, a)$

Refer to Fig. 6. In this case, the partial color sums of v_4, v_6, and v_7 are all equal to $2a$. Since c is a sigma coloring, the actual color sums of the three vertices must be different from each other. This implies that $c(v_2)$, $c(v_8)$, and $c(u_1)$ must all have different values. Clearly, this is not possible since there are only two colors.

Case 2 $\big(c(v_4), c(v_6), c(v_7)\big) = (b, a, a)$

In this case, the partial color sums of v_6 and v_7 are both equal to $a + b$. Since c is a sigma coloring, the actual color sums of v_6 and v_7 must be distinct. Hence, $c(u_1) \neq c(v_8)$.

(2.a) Suppose $\big(c(u_1), c(v_8)\big) = (a, b)$. Refer to Fig. 7. In this case, $\sigma(v_6) = 2a + b$ and $\sigma(v_7) = 2b+a$. Since $\sigma(v_6)$ cannot be equal to $\sigma(u_1)$, we must have $\{c(u_2), c(u_3)\} \neq \{a, b\}$. Moreover, Case 1 implies that $\big(c(u_2), c(u_3)\big) \neq (a, a)$. Hence, $\big(c(u_2), c(u_3)\big)$ must be (b, b). This implies that $\sigma(u_1) = 2b + a$ and that u_2 and u_3 both have partial color sums equal to $a + b$. Since the actual color sums of u_2 and u_3 cannot be equal to $\sigma(u_1)$, there are no possible values for $c(u_4)$ and $c(u_5)$ that do not violate the assumption that c is a sigma coloring.

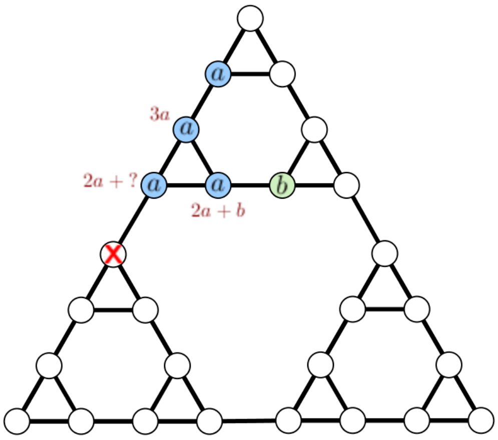

Figure 6. Case 1.

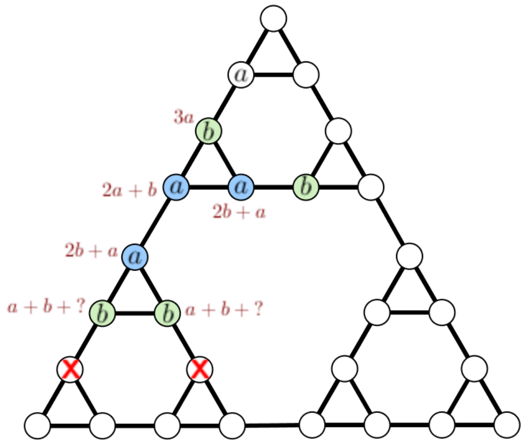

Figure 7. Case 2.a.

(2.b) Suppose $\big(c(u_1), c(v_8)\big) = (b, a)$. This case can be treated in a similar way as Case 2.1.

Case 3 $\big(c(v_4), c(v_6), c(v_7)\big) = (a, a, b)$

In this case, the partial color sums of v_4 and v_6 are both equal to $a + b$. Since c is a sigma coloring, the actual color sums of v_4 and v_6 must be distinct. Hence, $c(v_2) \neq c(u_1)$.

(3.a) Suppose $\big(c(v_2), c(u_1)\big) = (b, a)$. Then $\sigma(v_4) = 2b + a$ and $\sigma(v_6) = 2a + b$. This proceeds in the same manner as Case (2.a).

(3.b) Suppose $\big(c(v_2), c(u_1)\big) = (a, b)$. Then $\sigma(v_6) = 2b + a$ and $\sigma(v_4) = 2a + b$. It follows that $c(v_8) = a$ and $\sigma(v_7) = 3a$.

(i) Suppose $\big(c(v_5), c(v_9)\big) = (a, a)$. By Case 1, this leads to a contradiction.

(ii) Suppose $\big(c(v_5), c(v_9)\big) = (b, a)$. By Case 2, this leads to a contradiction as well.

(iii) Suppose $\big(c(v_5), c(v_9)\big) = (a, b)$. Refer to Fig. 8. In this case, the partial color sums of v_2 and v_3 are both equal to $2a$. Since their actual color sums must be distinct, there is no possible value for $c(v_1)$.

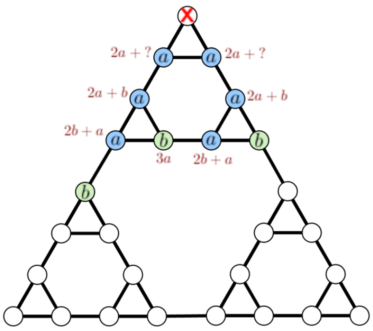

Figure 8. Case 3.b.iii.

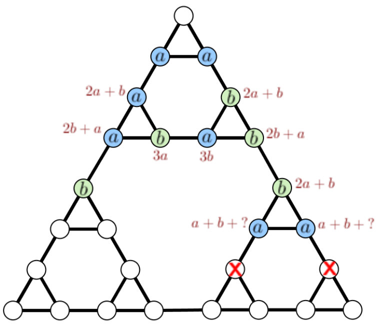

Figure 9. Case 3.b.iv.

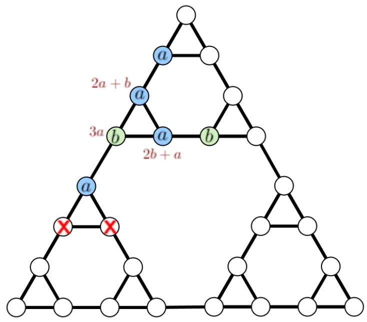

Figure 10. Case 4.

(iv) Suppose $\big(c(v_5), c(v_9)\big) = (b, b)$. If $c(v_3) = b$, then the partial color sums of v_2 and v_3 are equal and there is no possible value for $c(v_1)$. Hence, $c(v_3) = a$ and $\sigma(v_5) = 2a + b$. It follows that $c(w_1) = b$ and $\sigma(v_9) = 2b + a$. This case then proceeds similarly as Case (2.a). Refer to Fig. 9

Case 4 $\big(c(v_4), c(v_6), c(v_7)\big) = (a, b, a)$

Refer to Fig. 10. In this case, the partial color sums of v_4 and v_7 are both equal to $a+b$. Since c is a sigma coloring, the actual color sums of v_4 and v_7 must be distinct. Hence, $c(v_2) \neq c(v_8)$. This implies that $\{\sigma(v_4), \sigma(v_7)\} = \{2a + b, 2b + a\}$; hence, $c(u_1) = a$ so that $\sigma(v_6) = 3a$. By Cases 1, 2, and 3, there are no possible values for $c(u_2)$ and $c(u_3)$. Therefore, there is no sigma 2-coloring of H_3. Since $\chi(H_3) = 3$ and $\sigma(H_3) \leq \chi(H_3)$, we must have $\sigma(H_3) = 3$. □

3. The Sigma Chromatic Number of the Sierpiński Gasket Graphs

In this section, we prove that the sigma chromatic number of the Sierpiński gasket graph S_n is 2 for all $n \geq 2$. Throughout this section, we use a, b to denote distinct positive integers for which the sets $\{a + b, 2a, 2b\}$ and $\{2a + 2b, 3a + b, 3b + a, 4a, 4b\}$ are disjoint. In Fig. 11, we present a sigma 2-coloring of S_2 using a and b as colors. Our proof consists of the following steps:

(i) Construct a sigma 2-coloring of S_3 using the sigma 2-coloring of S_2 in Fig. 11
(ii) Construct a sigma 2-coloring of S_4 using the obtained sigma 2-coloring of S_3
(iii) Generalize to any S_n by induction

3.1. Construction of a sigma 2-coloring of S_3

Denote the coloring in Fig. 11 by LL. Perform two operations on LL to obtain two new colorings LR and U:

(i) To obtain LR, we rotate LL 120° clockwise. LR is shown in Fig. 12.
(ii) To obtain U, we reflect LL along the angle bisector of its lower left corner. We then interchange the colors a and b. U is shown in Fig. 13.

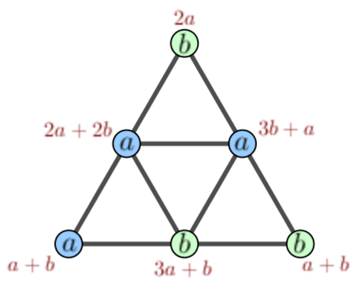

Figure 11. LL.

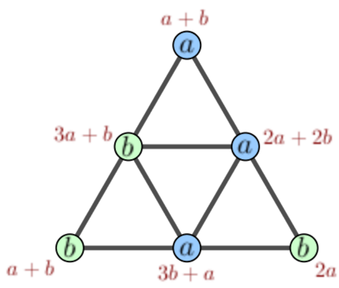

Figure 12. LR.

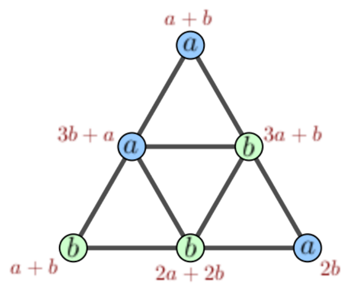

Figure 13. U.

Notice that the above operations ensure that the colorings in LR and U are sigma colorings as well. Using LL, LR, and U, construct a sigma coloring of S_3 as shown in Fig. 14. In this figure, a double arrow between a pair of corner vertices signifies that these two vertices are to be identified with each other. The resulting sigma coloring of S_3 is shown in Fig. 15.

3.2. Construction of a sigma 2-coloring of S_4

Denote by LL_2 the coloring in Fig. 15. As in the previous construction, perform operations on LL_2 to obtain two new colorings LR_2 and U_2:

(i) To obtain LR_2, reflect LL_2 along the angle bisector of its lower left corner, then interchange the colors a and b.
(ii) To obtain U_2, rotate LL 120° counterclockwise.

Using $\mathsf{LL}_2, \mathsf{U}_2$, and LR_2, construct the sigma 2-coloring of S_4 as done for S_3. The constructed sigma coloring is shown in Fig. 16.

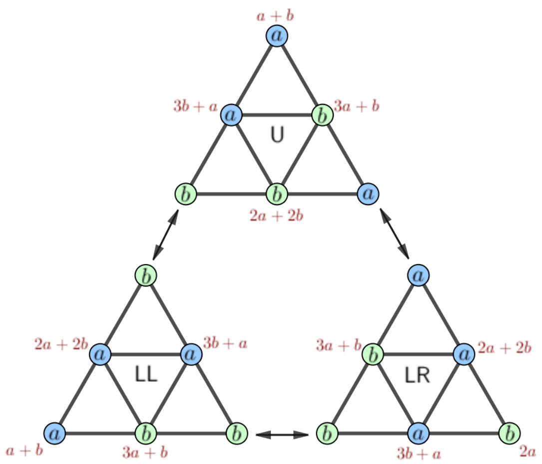

Figure 14. Constructing a sigma coloring of S_3 using LL, LR, and U.

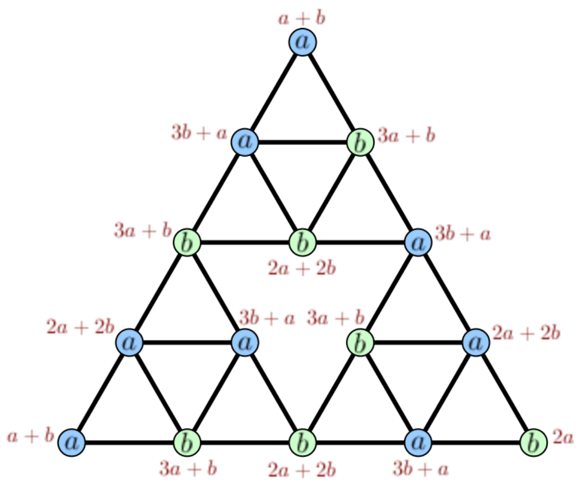

Figure 15. The constructed sigma coloring of S_3.

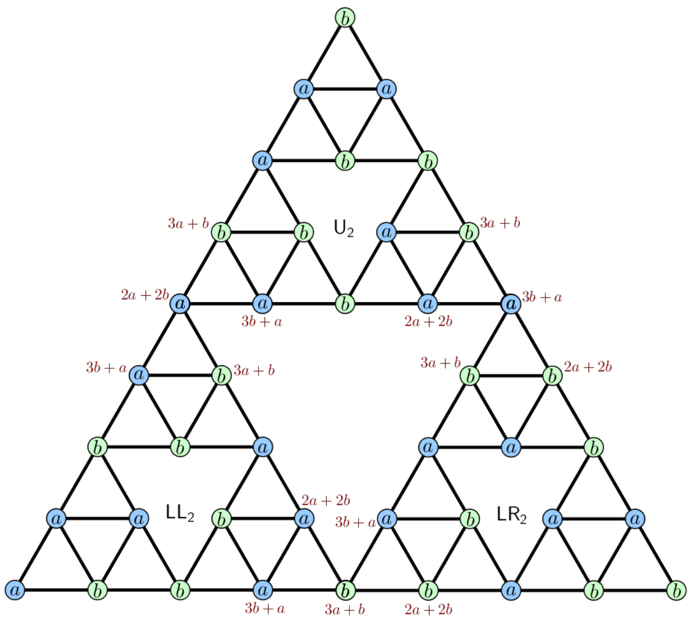

Figure 16. The constructed sigma 2-coloring of S_4.

3.3. Induction

In Fig. 17, a simplified version of Fig. 16 is shown. This version shows only the vertices of the three corner triangles, their colors, and their color sums.

Fig. 16 shows the important similarities between the sigma 2-colorings of S_2 and S_4:

- Corresponding corner vertices have the same colors and color sums.
- The corresponding neighbors of corresponding corner vertices have the same colors and color sums.

Due to these similarities, it is possible to repeat the operations performed previously to obtain sigma 2-colorings of S_5 and S_6. This process can then be repeated to complete the proof. Therefore, the sigma chromatic of S_n, $n \geq 2$, is 2. $\square$

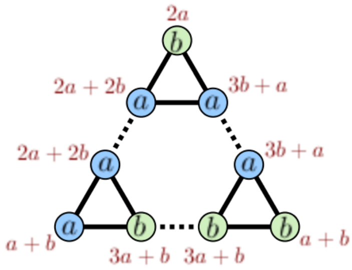

Figure 17. A simplified version of Fig. 16.

The preceding proof leads to the following result, which is more precise than statement (2) of Theorem 1.6.

Theorem 3.1. *Let $n \geq 2$ be an integer.*

(i) If n is even, then S_n has a sigma 2-coloring of the form shown in Fig. 18.

(ii) If n is odd, then S_n has a sigma 2-coloring of the form shown in Fig. 19.

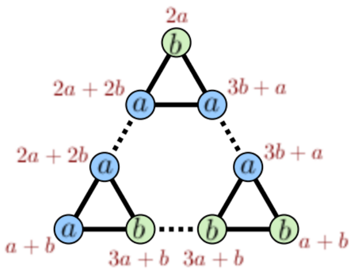

Figure 18. Sigma 2-coloring for S_n, n even.

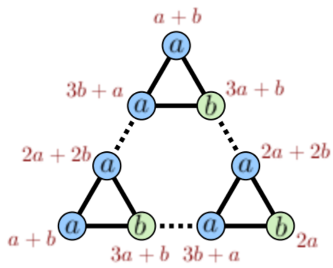

Figure 19. Sigma 2-coloring for S_n, n odd.

4. Uniqueness of the Sigma 2-coloring of Sierpiński Gasket Graphs

As in Section 3, use a, b to denote distinct positive integers for which the sets $\{a+b, 2a, 2b\}$ and $\{2a+2b, 3a+b, 3b+a, 4a, 4b\}$ are disjoint. We now make the following definitions.

Definition 4.1. *Let $n \geq 2$ be an integer and let c be a sigma 2-coloring, using the colors a and b, of S_n. Define the operations $r, s,$ and t as follows:*

(i) The coloring rc is obtained by rotating c $120°$ clockwise.

(ii) The coloring sc is obtained by reflecting c along the angle bisector of the lower left corner vertex.

(iii) The coloring tc is obtained by interchanging the colors a and b.

Also, define G to be the group $\langle r, s, t\rangle$ and Gc to be the set $\{gc : g \in G\}$.

Clearly, G is isomorphic to $D_3 \times \mathbb{Z}_2$; hence, it has 12 elements. Moreover, all the elements of Gc are also sigma 2-colorings of S_n.

Definition 4.2. *Let $n \geq 2$ be an integer and let c be a sigma 2-coloring of S_n. Define the sigma 2-colorings $\mathsf{LL}(c), \mathsf{U}(c), \mathsf{LR}(c)$ of S_{n-1} as follows:*

(i) $\mathsf{LL}(c)$ is the restriction of c to the lower left S_{n-1} subgraph of S_n.

(ii) $\mathsf{U}(c)$ *is the restriction of* c *to the upper* S_{n-1} *subgraph of* S_n.

(iii) $\mathsf{LR}(c)$ *is the restriction of* c *to the lower right* S_{n-1} *subgraph of* S_n.

We also write $c = (\alpha, \beta, \gamma)$ to signify that α, β, and γ are sigma 2-colorings of S_{n-1} for which $(\mathsf{LL}(c), \mathsf{U}(c), \mathsf{LR}(c)) = (\alpha, \beta, \gamma)$. For instance, the coloring c_3 shown in Fig. 15 can be written as $c_3 = (c_2, tsc_2, rc_2)$, where c_2 is the coloring shown in Fig. 11.

For S_n, $n \geq 2$, we denote by c_n the sigma 2-coloring constructed in the proof in Section 3. Recall that c_n follows the forms shown in Figures 18 and 19. In order to prove statement (3) of Theorem 1.6, we prove the following equivalent statement.

Lemma 4.3. *Let* a, b *be fixed distinct positive integers for which the sets* $\{a+b, 2a, 2b\}$ *and* $\{2a+2b, 3a+b, 3b+a, 4a, 4b\}$ *are disjoint. Then any sigma 2-coloring* c *of* S_n, $n \geq 2$, *is in* Gc_n.

Proof. Our proof is by induction. The case $n = 2$ can be verified easily; that is, the totality of sigma 2-colorings of S_2 is given by Gc_2, where c_2 is the sigma 2-coloring shown in Fig. 11.

Now, suppose the statement holds for n and let c be a sigma 2-coloring of S_{n+1}. By the assumption on the colors a and b, $\mathsf{LL}(c), \mathsf{U}(c)$, and $\mathsf{LR}(c)$ must be sigma 2-colorings of S_n; hence, by the inductive hypothesis, they belong to the set Gc_n.

Case 1. Suppose n is even.

By applying t to c at most once, we can guarantee that the resulting coloring c' has the lower left corner vertex colored a; that is, $\mathsf{LL}(c')$ is in $\{c_n, sc_n, trc_n, strc_n, tr^2c_n, str^2c_n\}$. Moreover, by applying s to c' at most once, we can guarantee that the resulting coloring c'' has $\mathsf{LL}(c'')$ that is in $\{c_n, trc_n, tr^2c_n\}$. Therefore, we can simply assume without loss of generality that $\mathsf{LL}(c)$ is in $\{c_n, trc_n, tr^2c_n\}$.

Case 1.1. Suppose $\mathsf{LL}(c) = c_n$. Since there are only 12 possible choices each for $\mathsf{U}(c)$ and $\mathsf{LR}(c)$, it is easy to verify that $c = (c_n, tsc_n, rc_n)$. In this case, $c = c_{n+1}$, which is the sigma 2-coloring constructed in Section 3. Naturally, $c = c_{n+1} \in Gc_{n+1}$.

Case 1.2. Suppose $\mathsf{LL}(c) = trc_n$. In this case, we must have $c = (trc_n, sr^2c_n, sc_n)$. Furthermore, we have

$$c_{n+1} = (c_n, tsc_n, rc_n) \Longrightarrow rsc_{n+1} = (trc_n, rsc_n, rsrc_n).$$

By applying the property that $(sr)^2$ is the identity, rsc_{n+1} simplifies to c; that is, $c \in Gc_{n+1}$.

Case 1.3. Suppose $\mathsf{LL}(c) = tr^2c_n$. In this case, we must have $c = (tr^2c_n, trc_n, sr^2c_n)$. Furthermore, we have

$$c_{n+1} = (c_n, tsc_n, rc_n) \Longrightarrow trc_{n+1} = (tr^2c_n, trc_n, rsc_n).$$

As in Case 1.2., trc_{n+1} simplifies to c; that is, $c \in Gc_{n+1}$.

Case 2. Suppose n is odd.

Similar to Case 1, we can simply assume without loss of generality that $\mathsf{LL}(c)$ is in $\{c_n, r^2c_n, trc_n\}$.

Case 2.1. Suppose $\mathsf{LL}(c) = c_n$. In this case, $c = (c_n, r^2c_n, tsc_n)$. In this case, $c = c_{n+1}$, which is the sigma 2-coloring constructed in Section 3. Naturally, $c = c_{n+1} \in Gc_{n+1}$.

Case 2.2. Suppose $\mathsf{LL}(c) = r^2c_n$. In this case, $c = (r^2c_n, tsc_n, tsrc_n)$. Furthermore,

$$c_{n+1} = (c_n, r^2c_n, tsc_n) \Longrightarrow srtc_{n+1} = (srsc_n, tsc_n, tsrc_n).$$

Hence, $srtc_{n+1}$ is equal to c; that is, $c \in Gc_{n+1}$.

Case 2.3. Suppose $\mathsf{LL}(c) = trc_n$. In this case, $c = (trc_n, src_n, tr^2c_n)$. Furthermore,

$$c_{n+1} = (c_n, r^2c_n, tsc_n) \Longrightarrow r^2tc_{n+1} = (trc_n, r^2sc_n, tr^2c_n).$$

Hence, r^2tc_{n+1} is equal to c; that is, $c \in Gc_{n+1}$.

This completes the proof of the lemma and, consequently, of statement (3) of Theorem 1.6. □

5. Conclusion

In this paper, we have determined the sigma chromatic numbers of two families of graphs: the Sierpiński gasket graphs and the Hanoi graphs. For the Hanoi graphs, our approach involved showing that, for $n \geq 3$, the sigma 2-colorability of H_{n+1} implies the sigma 2-colorability of H_n. On the other hand, for the Sierpiński gasket graphs, we employed a recursive construction of a sigma 2-coloring of S_n. Moreover, we have proven that this sigma 2-coloring is unique up to rotations, reflections, and choice of colors. This uniqueness result is the sigma coloring analog of Klavžar's uniqueness result for proper 3-colorings of S_n.

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