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Agnes Garciano
Reginaldo M. Marcelo
Mari-Jo P. Ruiz
Mark Anthony C. Tolentino

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The sigma chromatic number of the Sierpiński gasket graphs and the Hanoi graphs

A D Garciano, R M Marcelo, M J P Ruiz and M A C Tolentino
Department of Mathematics, School of Science and Engineering, Loyola Schools, Ateneo de Manila University, Philippines
E-mail: agarciano@ateneo.edu, rmarcelo@ateneo.edu, mruiz@ateneo.edu, mtolentino@ateneo.edu

Abstract. A vertex coloring \( c : V(G) \to \mathbb{N} \) of a non-trivial connected graph \( G \) is called a sigma coloring if \( \sigma(u) \neq \sigma(v) \) for any pair of adjacent vertices \( u \) and \( v \). Here, \( \sigma(x) \) denotes the sum of the colors assigned to vertices adjacent to \( x \). The sigma chromatic number of \( G \), denoted by \( \sigma(G) \), is defined as the fewest number of colors needed to construct a sigma coloring of \( G \). In this paper, we determine the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs. Moreover, we prove the uniqueness of the sigma coloring for Sierpiński gasket graphs.

1. Introduction
In [4], Chartrand, Okamoto, and Zhang introduced a new kind of vertex coloring called a sigma coloring. It is defined as follows.

Definition 1.1 (Chartrand, Okamoto, Zhang [4]). For a non-trivial connected graph \( G \), let \( c : V(G) \to \mathbb{N} \) be a vertex coloring of \( G \). For each \( v \in V(G) \), the color sum of \( v \), denoted by \( \sigma(v) \), is defined to be the sum of the colors of the vertices adjacent to \( v \). If \( \sigma(u) \neq \sigma(v) \) for every two adjacent \( u, v \in V(G) \), then \( c \) is called a sigma coloring of \( G \). The minimum number of colors required in a sigma coloring of \( G \) is called its sigma chromatic number and is denoted by \( \sigma(G) \).

Sigma coloring is an example of a neighbor-distinguishing coloring, the most studied example of which is the proper vertex coloring. Over the years, various neighbor-distinguishing colorings have also been introduced and discussed in literature such as in [3] and [5]. The notion of sigma coloring is closely related to the vertex colorings/labellings, discussed in [1], [9], [12] that also use the sum of the colors/labels of a vertex’s neighbors.

Sigma colorings of different families of graphs have already been studied. For instance, Chartrand et al. determined the sigma chromatic numbers of paths, cycles, bipartite, and complete multipartite graphs in [4]. More recently, in [10], Luzon, Ruiz, and Tolentino have determined the sigma chromatic numbers of some families of circulant graphs. The complexity of the sigma coloring problem has also been studied in [6].

In this paper, we determine the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs.

The Sierpiński gasket graph \( S_n, n \geq 1 \), is the graph whose vertices are the intersection points of the finite Sierpiński gasket and whose edges are the line segments of the gasket [13].
An alternative description of $S_n$ can also be found in [8]. The construction of $S_n$ is shown in Fig. 1 while a sigma coloring of $S_2$ is shown in Fig. 2.

![Sierpinski gasket graphs](image1)

**Figure 1.** Sierpinski gasket graphs.

![Sigma coloring of $S_2$](image2)

**Figure 2.** A sigma coloring of $S_2$.

On the other hand, the Hanoi graphs ([2], [7], [8]) are derived from the states of the Tower of Hanoi problem. Like the Sierpinski gasket graphs, the Hanoi graphs are also constructed in an iterative manner as shown in Fig. 3. A sigma 2-coloring of $H_2$ is shown in Fig. 4.

![Hanoi graphs](image3)

**Figure 3.** Hanoi graphs.

The (ordinary) chromatic numbers of $S_n$ and $H_n$ have already been determined by other researchers. For the Hanoi graphs, Parisse has constructed a natural coloring that leads to the following result.

**Theorem 1.2** (Parisse, [11]). For all $n \geq 1$, the chromatic number of $H_n$ is 3.

For the Sierpinski gasket graphs, Teguia and Godbole proved the following:
Figure 4. A sigma 2-coloring of $H_2$.

**Theorem 1.3** (Teguia, Godbole [13]). *For all $n \geq 1$, the chromatic number of $S_n$ is 3.*

Moreover, Klavžar proved the following stronger result.

**Theorem 1.4** (Klavžar [8]). *The proper 3-coloring of $S_n$ is unique for any $n \geq 1.*

More precisely, Theorem 1.4 states that if $c_1$ and $c_2$ are two distinct proper 3-colorings of $S_n$, then it is possible to transform $c_1$ to $c_2$ by performing rotations, reflections, and/or a change of colors used.

In [4], Chartrand et al. also showed the following:

**Theorem 1.5** (Chartrand, Okamoto, Zhang [4]). *For every graph $G$,\[
\sigma(G) \leq \chi(G).
\]

Hence, the sigma chromatic numbers of the Sierpiński gasket graphs and the Hanoi graphs are at most three. In this paper, we determine the exact sigma chromatic number of these graphs. Moreover, we also prove the uniqueness, in the same sense as in Theorem 1.4, of the sigma coloring of $S_n$ ($n \geq 2$) that uses the minimum number of colors. The following are our main results.

**Theorem 1.6.**

1. For all $n \geq 3$, the sigma chromatic number of $H_n$ is 3.
2. For all $n \geq 2$, the sigma chromatic number of $S_n$ is 2.
3. For all $n \geq 2$, the sigma 2-coloring of $S_n$ is unique.

In Section 2, we prove statement (1) by showing that $H_n$, $n \geq 3$, does not have a sigma 2-coloring. In Section 3, we prove statement (2) by constructing iteratively a sigma 2-coloring of $S_n$, $n \geq 2$. Finally, in Section 4, we prove the uniqueness of this sigma 2-coloring.

2. The Sigma Chromatic Number of the Hanoi Graphs

In this section, we prove that the sigma chromatic number of the Hanoi graph $H_n$ is 3 for all $n \geq 3$. We begin with the following proposition.

**Proposition 2.1.** Let $n \geq 3$. If $H_n$ is not sigma 2-colorable, then neither is $H_{n+1}$.

**Proof.** Suppose that $H_{n+1}$ has a sigma 2-coloring $c$ that uses the colors $a$ and $b$. Moreover, assume that $a$ and $b$ have been chosen so that the sets $\{3a, 2a + b, 2b + a, 3b\}$ and $\{2a, a + b, 2b\}$ are disjoint.

Let $S$ be one of the three $H_n$ subgraphs of $H_{n+1}$ and define the coloring $c'$ to be the restriction of $c$ to $S$. We want to show that $c'$ is a sigma 2-coloring of $S$. 
To this end, let \( x \) and \( y \) be the two corner vertices of \( S \) that originally have degree three as vertices of \( H_{n+1} \). Note that \( c \) and \( c' \) induce the same color sums for all vertices in \( S \setminus \{x, y\} \). Hence, to show that \( c' \) is a sigma coloring of \( S \), we need to focus only on the color sums of \( x \) and \( y \). Now, since \( x \) and \( y \) are vertices with degree two in \( S \), their color sums with respect to \( c' \) are in the set \( \{2a, a + b, 2b\} \). Hence, by the assumption, these color sums cannot be equal to the color sum of any of the neighbors of \( x \) or \( y \) since these neighbors have degree three. This completes the proof.

Thus, to prove that \( \sigma(H_n) = 3, n \geq 3 \), it is sufficient to prove that \( H_3 \) is not sigma 2-colorable. We show this in the following lemma.

**Lemma 2.2.** The sigma chromatic number of \( H_3 \) is 3.

**Proof.** Suppose there is a sigma 2-coloring \( c : V(H_3) \to \{a, b\} \) of \( H_3 \). We show that \( c \) produces a contradiction. Let us label the vertices of \( H_3 \) as shown in Fig. 5.

![Figure 5. Vertex labels of the Hanoi graph \( H_3 \).](image)

We consider cases based on the possible values of \( c(v_4), c(v_6), \) and \( c(v_7) \). There are four cases, in each of which we show how the contradiction arises.

**Case 1** \( (c(v_4), c(v_6), c(v_7)) = (a, a, a) \)

Refer to Fig. 6. In this case, the partial color sums of \( v_4, v_6, \) and \( v_7 \) are all equal to \( 2a \). Since \( c \) is a sigma coloring, the actual color sums of the three vertices must be different from each other. This implies that \( c(v_2), c(v_8), \) and \( c(u_1) \) must all have different values. Clearly, this is not possible since there are only two colors.

**Case 2** \( (c(v_4), c(v_6), c(v_7)) = (b, a, a) \)

In this case, the partial color sums of \( v_6 \) and \( v_7 \) are both equal to \( a + b \). Since \( c \) is a sigma coloring, the actual color sums of \( v_6 \) and \( v_7 \) must be distinct. Hence, \( c(u_1) \neq c(v_6). \)

(2.a) Suppose \( (c(u_1), c(v_8)) = (a, b) \). Refer to Fig. 7. In this case, \( \sigma(v_6) = 2a + b \) and \( \sigma(v_7) = 2b + a \). Since \( \sigma(v_6) \) cannot be equal to \( \sigma(u_1) \), we must have \( \{c(u_2), c(v_3)\} \neq \{a, b\} \). Moreover, Case 1 implies that \( (c(u_2), c(u_3)) \neq (a, a) \). Hence, \( (c(u_2), c(u_3)) \) must be \( (b, b) \). This implies that \( \sigma(u_1) = 2b + a \) and that \( u_2 \) and \( u_3 \) both have partial color sums equal to \( a + b \). Since the actual color sums of \( u_2 \) and \( u_3 \) cannot be equal to \( \sigma(u_1) \), there are no possible values for \( c(u_4) \) and \( c(u_5) \) that do not violate the assumption that \( c \) is a sigma coloring.
Figure 6. Case 1.

Figure 7. Case 2.a.

(2.b) Suppose \((c(u_1), c(v_8)) = (b, a)\). This case can be treated in a similar way as Case 2.1.

Case 3 \((c(v_4), c(v_5), c(v_7)) = (a, a, b)\)

In this case, the partial color sums of \(v_4\) and \(v_6\) are both equal to \(a + b\). Since \(c\) is a sigma coloring, the actual color sums of \(v_4\) and \(v_6\) must be distinct. Hence, \(c(v_2) \neq c(u_1)\).

(3.a) Suppose \((c(v_2), c(u_1)) = (b, a)\). Then \(\sigma(v_4) = 2b + a\) and \(\sigma(v_6) = 2a + b\). This proceeds in the same manner as Case (2.a).

(3.b) Suppose \((c(v_2), c(u_1)) = (a, b)\). Then \(\sigma(v_4) = 2b + a\) and \(\sigma(v_6) = 2a + b\). It follows that \(c(v_8) = a\) and \(\sigma(v_7) = 3a\).

(i) Suppose \((c(v_3), c(v_9)) = (a, a)\). By Case 1, this leads to a contradiction.

(ii) Suppose \((c(v_5), c(v_9)) = (b, a)\). By Case 2, this leads to a contradiction as well.

(iii) Suppose \((c(v_5), c(v_9)) = (a, b)\). Refer to Fig. 8. In this case, the partial color sums of \(v_2\) and \(v_3\) are both equal to \(2a\). Since their actual color sums must be distinct, there is no possible value for \(c(v_1)\).

Figure 8. Case 3.b.iii.

Figure 9. Case 3.b.iv.

(iii) Suppose \((c(v_5), c(v_9)) = (b, b)\). If \(c(v_3) = b\), then the partial color sums of \(v_2\) and \(v_3\) are equal and there is no possible value for \(c(v_1)\). Hence, \(c(v_5) = a\) and \(\sigma(v_5) = 2a + b\). It follows that \(c(w_1) = b\) and \(\sigma(v_9) = 2b + a\). This case then proceeds similarly as Case (2.a). Refer to Fig. 9.
Case 4 \((c(v_4), c(v_6), c(v_7)) = (a, b, a)\)

Refer to Fig. 10. In this case, the partial color sums of \(v_4\) and \(v_7\) are both equal to \(a + b\). Since \(c\) is a sigma coloring, the actual color sums of \(v_4\) and \(v_7\) must be distinct. Hence, \(c(v_2) \neq c(v_8)\).

This implies that \(\{\sigma(v_4), \sigma(v_7)\} = \{2a + b, 2b + a\}\); hence, \(c(u_1) = a\) so that \(\sigma(v_6) = 3a\). By Cases 1, 2, and 3, there are no possible values for \(c(u_2)\) and \(c(u_3)\). Therefore, there is no sigma 2-coloring of \(H_3\). Since \(\chi(H_3) = 3\) and \(\sigma(H_3) \leq \chi(H_3)\), we must have \(\sigma(H_3) = 3\). □

3. The Sigma Chromatic Number of the Sierpiński Gasket Graphs

In this section, we prove that the sigma chromatic number of the Sierpiński gasket graph \(S_n\) is 2 for all \(n \geq 2\). Throughout this section, we use \(a, b\) to denote distinct positive integers for which the sets \(\{a + b, 2a, 2b\}\) and \(\{2a + 2b, 3a + b, 3b + a, 4a, 4b\}\) are disjoint. In Fig. 11, we present a sigma 2-coloring of \(S_2\) using \(a\) and \(b\) as colors. Our proof consists of the following steps:

(i) Construct a sigma 2-coloring of \(S_3\) using the sigma 2-coloring of \(S_2\) in Fig. 11

(ii) Construct a sigma 2-coloring of \(S_4\) using the obtained sigma 2-coloring of \(S_3\)

(iii) Generalize to any \(S_n\) by induction

3.1. Construction of a sigma 2-coloring of \(S_3\)

Denote the coloring in Fig. 11 by LL. Perform two operations on LL to obtain two new colorings LR and U:

(i) To obtain LR, we rotate LL \(120^\circ\) clockwise. LR is shown in Fig. 12.

(ii) To obtain U, we reflect LL along the angle bisector of its lower left corner. We then interchange the colors \(a\) and \(b\). U is shown in Fig. 13.

![Figure 11. LL.](image)

![Figure 12. LR.](image)

![Figure 13. U.](image)

Notice that the above operations ensure that the colorings in LR and U are sigma colorings as well. Using LL, LR, and U, construct a sigma coloring of \(S_3\) as shown in Fig. 14. In this figure, a double arrow between a pair of corner vertices signifies that these two vertices are to be identified with each other. The resulting sigma coloring of \(S_3\) is shown in Fig. 15.

3.2. Construction of a sigma 2-coloring of \(S_4\)

Denote by LL2 the coloring in Fig. 15. As in the previous construction, perform operations on LL2 to obtain two new colorings LR2 and U2:

(i) To obtain LR2, reflect LL2 along the angle bisector of its lower left corner, then interchange the colors \(a\) and \(b\).

(ii) To obtain U2, rotate LL \(120^\circ\) counterclockwise.

Using LL2, U2, and LR2, construct the sigma 2-coloring of \(S_4\) as done for \(S_3\). The constructed sigma coloring is shown in Fig. 16.
3.3. Induction

In Fig. 17, a simplified version of Fig. 16 is shown. This version shows only the vertices of the three corner triangles, their colors, and their color sums.

Fig. 16 shows the important similarities between the sigma 2-colorings of $S_2$ and $S_4$:

- Corresponding corner vertices have the same colors and color sums.
- The corresponding neighbors of corresponding corner vertices have the same colors and color sums.

Due to these similarities, it is possible to repeat the operations performed previously to obtain sigma 2-colorings of $S_5$ and $S_6$. This process can then be repeated to complete the proof. Therefore, the sigma chromatic of $S_n$, $n \geq 2$, is 2.

\[\square\]
The preceding proof leads to the following result, which is more precise than statement (2) of Theorem 1.6.

**Theorem 3.1.** Let \( n \geq 2 \) be an integer.

(i) If \( n \) is even, then \( S_n \) has a sigma 2-coloring of the form shown in Fig. 18.

(ii) If \( n \) is odd, then \( S_n \) has a sigma 2-coloring of the form shown in Fig. 19.

**Figure 17.** A simplified version of Fig. 16.

**Figure 18.** Sigma 2-coloring for \( S_n \), \( n \) even.

**Figure 19.** Sigma 2-coloring for \( S_n \), \( n \) odd.

4. **Uniqueness of the Sigma 2-coloring of Sierpiński Gasket Graphs**

As in Section 3, use \( a, b \) to denote distinct positive integers for which the sets \{\( a + b, 2a, 2b \)\} and \{\( 2a + 2b, 3a + b, 3b + a, 4a, 4b \)\} are disjoint. We now make the following definitions.

**Definition 4.1.** Let \( n \geq 2 \) be an integer and let \( c \) be a sigma 2-coloring, using the colors \( a \) and \( b \), of \( S_n \). Define the operations \( r, s, \) and \( t \) as follows:

(i) The coloring \( rc \) is obtained by rotating \( c \) 120° clockwise.

(ii) The coloring \( sc \) is obtained by reflecting \( c \) along the angle bisector of the lower left corner vertex.

(iii) The coloring \( tc \) is obtained by interchanging the colors \( a \) and \( b \).

Also, define \( G \) to be the group \( \langle r, s, t \rangle \) and \( G_c \) to be the set \( \{gc : g \in G\} \).

Clearly, \( G \) is isomorphic to \( D_3 \times \mathbb{Z}_2 \); hence, it has 12 elements. Moreover, all the elements of \( G_c \) are also sigma 2-colorings of \( S_n \).

**Definition 4.2.** Let \( n \geq 2 \) be an integer and let \( c \) be a sigma 2-coloring of \( S_n \). Define the sigma 2-colorings \( \text{LL}(c), \text{U}(c), \text{LR}(c) \) of \( S_{n-1} \) as follows:

(i) \( \text{LL}(c) \) is the restriction of \( c \) to the lower left \( S_{n-1} \) subgraph of \( S_n \).
(iii) \( U(c) \) is the restriction of \( c \) to the upper \( S_{n-1} \) subgraph of \( S_n \).

(iii) \( LR(c) \) is the restriction of \( c \) to the lower right \( S_{n-1} \) subgraph of \( S_n \).

We also write \( c = (\alpha, \beta, \gamma) \) to signify that \( \alpha, \beta, \) and \( \gamma \) are sigma 2-colorings of \( S_{n-1} \) for which \((LL(c), U(c), LR(c)) = (\alpha, \beta, \gamma)\). For instance, the coloring \( c_3 \) shown in Fig. 15 can be written as \( c_3 = (c_2, tsc_2, rrc_2) \), where \( c_2 \) is the coloring shown in Fig. 11.

For \( S_n, n \geq 2 \), we denote by \( c_n \) the sigma 2-coloring constructed in the proof in Section 3. Naturally, \( c\{n\} \) is the sigma 2-coloring of \( S_n \); that is, \( c_{n+1} \) follows the forms shown in Figures 18 and 19. In order to prove statement (3) of Theorem 1.6, we prove the following equivalent statement.

**Lemma 4.3.** Let \( a, b \) be fixed distinct positive integers for which the sets \( \{a + b, 2a, 2b\} \) and \( \{2a + 2b, 3a + b, 3b + a, 4a, 4b\} \) are disjoint. Then any sigma 2-coloring \( c \) of \( S_n, n \geq 2, \) is in \( Gc_n \).

**Proof.** Our proof is by induction. The case \( n = 2 \) can be verified easily; that is, the totality of sigma 2-colorings of \( S_2 \) is given by \( Gc_2 \), where \( c_2 \) is the sigma 2-coloring shown in Fig. 11.

Now, suppose the statement holds for \( n \) and let \( c \) be a sigma 2-coloring of \( S_{n+1} \). By the assumption on the colors \( a \) and \( b \), \( LL(c), U(c), \) and \( LR(c) \) must be sigma 2-colorings of \( S_n \); hence, by the inductive hypothesis, they belong to the set \( Gc_n \).

**Case 1.** Suppose \( n \) is even.

By applying \( t \) to \( c \) at most once, we can guarantee that the resulting coloring \( c' \) has the lower left corner vertex colored \( a \); that is, \( LL(c') \) is in \( \{c_n, sc_n, trc_n, strc_n, tr^2c_n, str^2c_n\} \). Moreover, by applying \( s \) to \( c' \) at most once, we can guarantee that the resulting coloring \( c'' \) has \( LL(c'') \) that is in \( \{c_n, trc_n, tr^2c_n\} \). Therefore, we can simply assume without loss of generality that \( LL(c) \) is in \( \{c_n, trc_n, tr^2c_n\} \).

**Case 1.1.** Suppose \( LL(c) = c_n \). Since there are only 12 possible choices each for \( U(c) \) and \( LR(c) \), it is easy to verify that \( c = (c_n, tsc_n, rrc_n) \). In this case, \( c = c_{n+1} \), which is the sigma 2-coloring constructed in Section 3. Naturally, \( c = c_{n+1} \) belongs to \( Gc_{n+1} \).

**Case 1.2.** Suppose \( LL(c) = trc_n \). In this case, we must have \( c = (trc_n, sr^2c_n, sc_n) \). Furthermore, we have \( c_{n+1} = (c_n, tsc_n, rrc_n) \).

By applying the property that \( (sr)^2 \) is the identity, \( rsc_{n+1} \) simplifies to \( c_n \); that is, \( c \in Gc_{n+1} \).

**Case 1.3.** Suppose \( LL(c) = tr^2c_n \). In this case, we must have \( c = (tr^2, trc_n, sr^2c_n) \). Furthermore, we have \( c_{n+1} = (c_n, tsc_n, rrc_n) \) and \( rsc_{n+1} = (sr^2c_n, tsc_n, trc_n) \).

As in Case 1.2., \( trc_{n+1} \) simplifies to \( c_n \); that is, \( c \in Gc_{n+1} \).

**Case 2.** Suppose \( n \) is odd.

Similar to Case 1, we can simply assume without loss of generality that \( LL(c) \) is in \( \{c_n, r^2c_n, trc_n\} \).

**Case 2.1.** Suppose \( LL(c) = c_n \). In this case, \( c = (c_n, r^2c_n, tsc_n) \). In this case, \( c = c_{n+1} \), which is the sigma 2-coloring constructed in Section 3. Naturally, \( c = c_{n+1} \) belongs to \( Gc_{n+1} \).

**Case 2.2.** Suppose \( LL(c) = r^2c_n \). In this case, \( c = (r^2c_n, tsc_n, trc_n) \). Furthermore, \( c_{n+1} = (c_n, r^2c_n, tsc_n) \).

Hence, \( sr^2c_{n+1} \) is equal to \( c_n \); that is, \( c \in Gc_{n+1} \).

**Case 2.3.** Suppose \( LL(c) = trc_n \). In this case, \( c = (trc_n, sr^2c_n, tr^2c_n) \). Furthermore, \( c_{n+1} = (c_n, r^2c_n, tsc_n) \).

Hence, \( r^2t_{n+1} \) is equal to \( c_n \); that is, \( c \in Gc_{n+1} \).

This completes the proof of the lemma and, consequently, of statement (3) of Theorem 1.6.
5. Conclusion
In this paper, we have determined the sigma chromatic numbers of two families of graphs: the Sierpiński gasket graphs and the Hanoi graphs. For the Hanoi graphs, our approach involved showing that, for \( n \geq 3 \), the sigma 2-colorability of \( H_{n+1} \) implies the sigma 2-colorability of \( H_n \). On the other hand, for the Sierpiński gasket graphs, we employed a recursive construction of a sigma 2-coloring of \( S_n \). Moreover, we have proven that this sigma 2-coloring is unique up to rotations, reflections, and choice of colors. This uniqueness result is the sigma coloring analog of Klavžar’s uniqueness result for proper 3-colorings of \( S_n \).

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