

Ateneo de Manila University

Archium Ateneo

Mathematics Faculty Publications

Mathematics Department

2015

A Simpler Proof for the ϵ - δ Characterization of Baire Class One Functions

Emmanuel A. Cabral

Ateneo de Manila University, ecabral@ateneo.edu

Jonald P. Fenecios

Follow this and additional works at: <https://archium.ateneo.edu/mathematics-faculty-pubs>



Part of the [Mathematics Commons](#)

Custom Citation

Fenecios, Jonald P.; Cabral, Emmanuel A. A Simpler Proof for the ϵ - δ Characterization of Baire Class One Functions. *Real Anal. Exchange* 39 (2014), no. 2, 441–446. <https://projecteuclid.org/euclid.rae/1435670005>

This Article is brought to you for free and open access by the Mathematics Department at Archium Ateneo. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of Archium Ateneo. For more information, please contact oadrcw.ls@ateneo.edu.

Jonald P. Fenecios, Department of Mathematics, Ateneo de Davao University, E. Jacinto Street, 8000 Davao City, The Philippines. email: jpfenecios@addu.edu.ph

Emmanuel A. Cabral, Department of Mathematics, Ateneo de Manila University, Loyola Heights, 1108 Quezon City, The Philippines. email: ecabral@ateneo.edu

A SIMPLER PROOF FOR THE ϵ - δ CHARACTERIZATION OF BAIRE CLASS ONE FUNCTIONS

Abstract

We offer a new and simpler proof of a recent ϵ - δ characterization of Baire class one functions using a theorem by Henri Lebesgue. The proof is more elementary in the sense that it does not use the Baire Category Theorem. Furthermore, the proof requires only that the domain and range be separable metric spaces instead of Polish spaces.

1 Introduction

Let X and Y be metric spaces. A function $f : X \rightarrow Y$ is Baire class one if for every open set U in Y , $f^{-1}(U)$ is F_σ . Henri Lebesgue proved in 1904 the real line version of the following theorem:

Theorem 1. ([2, p. 115], [4, p. 375]) *Let Y be a separable metric space. A function $f : X \rightarrow Y$ is Baire class one if and only if for each natural number k , there exists a sequence of closed sets $\{E_n\}$ in X such that $X = \bigcup_{n=1}^{+\infty} E_n$ and $\omega_f(E_n) < \frac{1}{k}$ for each n where*

$$\omega_f(E_n) = \sup \{d_Y(f(x), f(y)) : x, y \in E_n\}$$

Mathematical Reviews subject classification: Primary: 26A21

Key words: Baire class one function, oscillation of a function, Polish spaces, separable metric spaces

Received by the editors July 18, 2013

Communicated by: Miroslav Zeleny

denotes the oscillation of f on E_n .

For easy reference, we shall call Theorem 1 Lebesgue's theorem.

More than a hundred years later since Rene Baire [1] defined this class of functions, a new characterization of Baire class one functions in terms of ϵ - δ was discovered by P.Y. Lee, W.K. Tang and D. Zhao [6] and later independently by D.N. Sarkhel [7]. We state it as a theorem below:

Theorem 2. *Let (X, d_X) and (Y, d_Y) be Polish spaces. The following statements are equivalent.*

- (1) $f : X \rightarrow Y$ is Baire class one
- (2) For each $\epsilon > 0$ there is a positive function $\delta : X \rightarrow \mathbb{R}^+$ such that for any $x, y \in X$

$$d_X(x, y) < \min\{\delta(x), \delta(y)\} \implies d_Y(f(x), f(y)) < \epsilon.$$

The proof of Theorem 2 relies on the Baire Category Theorem as well as on the fact that a function $f : X \rightarrow Y$ is Baire class one if and only if for every closed K in X , $f|_K$ has at least one point of continuity in K . In this paper, we shall prove Theorem 2 in a more general setting using Lebesgue's theorem.

2 A New Proof

Throughout the paper, (X, d_X) and (Y, d_Y) are assumed to be separable metric spaces. Denote the minimum between a and b by $a \wedge b$, the closure of a set A by \overline{A} and its diameter by $\text{diam}\{A\}$. Also denote the open ball with center $x_0 \in X$ and radius $\delta > 0$ by $N_\delta(x_0)$, that is, $N_\delta(x_0) = \{y \in X : d_X(x_0, y) < \delta\}$.

Before we provide the proof of our main theorem, let us recall first the following important results. Proposition 3 is stated for the real number line in [8, Lemma 1] without proof. On the other hand, Lemma 4 is proved in [8, Lemma 2] for the real line. Though Proposition 3 is quite well-known it is hard to find a proof for general spaces in the literature. For the sake of completeness, we shall give proofs of Proposition 3 and Lemma 4 in space X by adapting the proofs found in [3, p. 75] and [8, Lemma 2], respectively.

Proposition 3. *If $X = \bigcup_{n=1}^{+\infty} E_n$ with each E_n an F_σ set in X , then there are disjoint F_σ sets F_n , $n = 1, 2, \dots$, in X such that $F_n \subseteq E_n$ and $X = \bigcup_{n=1}^{+\infty} F_n$.*

PROOF. For every n , $E_n = \bigcup_{i=1}^{+\infty} E_n^i$ for some sequence $\{E_n^i\}$ of closed sets in X . Thus, we can express $X = \bigcup_{k=1}^{+\infty} A_k$ where for each k , A_k is closed in X and $A_k \subseteq E_n$ for some n . Let $H_1 = A_1$ and $H_k = A_k - (A_1 \cup A_2 \cup \cdots \cup A_{k-1})$ for $k \geq 2$. Notice that each H_k is F_σ in X , $X = \bigcup_{k=1}^{+\infty} H_k$ and $H_i \cap H_j = \emptyset$ for $i \neq j$. For each i , let $N_i = \{k \in \mathbb{N} : H_k \subseteq E_i\}$. The sets $F_i = \bigcup_{k \in N_i} H_k$ are pairwise disjoint F_σ sets, $F_i \subseteq E_i$ for each i and $X = \bigcup_{i=1}^{+\infty} F_i$. \square

Lemma 4. Let $X = \bigcup_{n=1}^{+\infty} F_n$ where F_n 's are disjoint F_σ sets. Then there is a positive function $\delta(\cdot)$ on X such that $x \in F_n$, $y \in F_m$ and $n \neq m$ imply

$$d_X(x, y) \geq \delta(x) \wedge \delta(y).$$

PROOF. Since each F_n is F_σ set, then there exists a sequence $\{F_n^i\}_{i=1}^{+\infty}$ of closed sets such that $F_n = \bigcup_{i=1}^{+\infty} F_n^i$ and $F_n^i \subseteq F_n^{i+1}$ for all i . For each n and $x \in F_n$, let $x \in F_n^{i_x}$, but $x \notin F_n^j$ for all $j < i_x$. Define a function $\delta : X \rightarrow \mathbb{R}^+$ such that

$$F_m^j \cap N_{\delta(x)}(x) = \emptyset$$

for all $m \neq n$ and $m + j \leq n + i_x$. Let $x \in F_n$ and $y \in F_m$, $m \neq n$. If $n + i_x \leq m + i_y$, then $F_n^{i_x} \cap N_{\delta(y)}(y) = \emptyset$. Hence, $d_X(x, y) \geq \delta(y)$. On the other hand, if $m + i_y \leq n + i_x$, then $F_m^{i_y} \cap N_{\delta(x)}(x) = \emptyset$. Thus, $d_X(x, y) \geq \delta(x)$. All these show that $d_X(x, y) \geq \delta(x) \wedge \delta(y)$. The lemma follows. \square

We are now ready to state and prove our main theorem. Recall that X and Y are assumed only to be separable metric spaces. Hence, our result generalizes the theorem of P.Y. Lee, W.K. Tang and D. Zhao [6].

Theorem 5. Let X and Y be separable metric spaces. The following statements are equivalent.

- (1) $f : X \rightarrow Y$ is Baire class one

(2) For each $\epsilon > 0$ there is a positive function $\delta : X \rightarrow \mathbb{R}^+$ such that for any $x, y \in X$

$$d_X(x, y) < \delta(x) \wedge \delta(y) \implies d_Y(f(x), f(y)) < \epsilon.$$

PROOF. (1) \Rightarrow (2). Suppose $f : X \rightarrow Y$ is Baire class one. Let $\epsilon > 0$ be given. Find a natural number k such that $\frac{1}{k} < \epsilon$. By Lebesgue's theorem, there exists a sequence of closed sets $\{E_n\}_{n=1}^{+\infty}$ in X such that $X = \bigcup_{n=1}^{+\infty} E_n$ and $\omega_f(E_n) < \frac{1}{k}$ for each n . There exists a sequence of F_σ sets $\{F_n\}_{n=1}^{+\infty}$ in X such that $X = \bigcup_{n=1}^{+\infty} F_n$, $F_n \subseteq E_n$ for each n and $F_i \cap F_j = \emptyset$ for $i \neq j$. By Lemma 4, there is a positive function $\delta : X \rightarrow \mathbb{R}^+$ such that if $x \in F_m$ and $y \in F_n$ with $m \neq n$ implies

$$d_X(x, y) \geq \delta(x) \wedge \delta(y).$$

Let $x, y \in X$ and $d_X(x, y) < \delta(x) \wedge \delta(y)$. By the property of δ there is a unique n such that $x, y \in F_n$. Since $F_n \subseteq E_n$ and $\omega_f(E_n) < \frac{1}{k}$ it immediately follows that $d_Y(f(x), f(y)) < \frac{1}{k} < \epsilon$.

(2) \Rightarrow (1) This direction is proved using the ideas from [7].

Suppose for each $\epsilon > 0$ there is a positive function $\delta : X \rightarrow \mathbb{R}^+$ such that for any $x, y \in X$

$$d_X(x, y) < \delta(x) \wedge \delta(y) \implies d_Y(f(x), f(y)) < \frac{\epsilon}{3}.$$

For each n , let $A_n = \{x \in X : \delta(x) > \frac{1}{n}\}$ and find a closed cover $\{F_n^k\}_{k=1}^{+\infty}$ of X such that $\text{diam } \{F_n^k\} < \frac{1}{n}$ for each k . Notice that $\{F_n^k\}_{k=1}^{+\infty}$ exists because X is a separable metric space. Hence, we can write the space X as

$$X = \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{+\infty} (F_n^k \cap \overline{A_n}).$$

Let $n, k \in \mathbb{N}$ and $x, y \in F_n^k \cap \overline{A_n}$. Since $\{A_n\}$ is an increasing sequence of sets then there is a $j > n$ such that $x, y \in A_j$. We can find $x_1, y_1 \in A_n$ such that

$$0 \leq d_X(x, x_1) < \frac{1}{j}, \quad 0 \leq d_X(y, y_1) < \frac{1}{j} \quad \text{and} \quad d_X(x_1, y_1) < \frac{1}{n}.$$

Since $A_n \subseteq A_j$, then $x_1, y_1 \in A_j$. Consequently,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_1)) + d_Y(f(x_1), f(y_1)) + d_Y(f(y), f(y_1)) < \epsilon.$$

This implies the Lebesgue's theorem. The proof is complete. \square

Acknowledgement: The authors wish to thank the referee for the suggestions that led to the significant improvement of this paper.

References

- [1] R. Baire, *Sur les fonctions des variables reelles*, Ann. Mat. Pura ed Appl. **3** (1899), 1–122.
- [2] D. M. Bressoud, *A Radical Approach To Lebesgue's Theory of Integration*, Cambridge University Press, United States of America, 2008.
- [3] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*, **4**, American Mathematical Society, 1994.
- [4] K. Kuratowski, *Topology*, Academic Press, London, 1966.
- [5] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, *Real Analysis*, Prentice-Hall, United States of America, 2001.
- [6] P. Y. Lee, W. K. Tang and D. Zhao, *An Equivalent definition of Functions of the first Baire class*, Proc. Amer. Math. Soc., **129**(8) (2001), 2273–2275.
- [7] D. N. Sarkhel, *Baire One Functions*, Bull. Inst. Math. Acad. Sinica, **31**(2) (2003), 143–149.
- [8] D. Zhao, *Functions Whose Composition With Baire Class One Functions Are Baire Class One*, Soochow J. Math., **33**(4) (2007), 543–551.

