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Abstract

We offer a new and simpler proof of a recent ϵ-δ characterization of Baire class one functions using a theorem by Henri Lebesgue. The proof is more elementary in the sense that it does not use the Baire Category Theorem. Furthermore, the proof requires only that the domain and range be separable metric spaces instead of Polish spaces.

1 Introduction

Let X and Y be metric spaces. A function $f : X \to Y$ is Baire class one if for every open set U in Y, $f^{-1}(U)$ is F_σ. Henri Lebesgue proved in 1904 the real line version of the following theorem:

Theorem 1. *([2, p. 115], [4, p. 375]) Let Y be a separable metric space. A function $f : X \to Y$ is Baire class one if and only if for each natural number k, there exists a sequence of closed sets $\{E_n\}$ in X such that $X = \bigcup_{n=1}^{+\infty} E_n$ and $\omega_f(E_n) < \frac{1}{k}$ for each n where*

$$\omega_f(E_n) = \sup\{d_Y(f(x), f(y)) : x, y \in E_n\}$$

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denotes the oscillation of f on E_n.

For easy reference, we shall call Theorem 1 Lebesgue's theorem.

More than a hundred years later since Rene Baire [1] defined this class of functions, a new characterization of Baire class one functions in terms of ϵ-δ was discovered by P.Y. Lee, W.K. Tang and D. Zhao [6] and later independently by D.N. Sarkhel [7]. We state it as a theorem below:

Theorem 2. *Let (X, d_X) and (Y, d_Y) be Polish spaces. The following statements are equivalent.*

(1) $f : X \to Y$ is Baire class one

(2) For each $\epsilon > 0$ there is a positive function $\delta : X \to \mathbb{R}^+$ such that for any $x, y \in X$

$$d_X(x, y) < \min\{\delta(x), \delta(y)\} \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

The proof of Theorem 2 relies on the Baire Category Theorem as well as on the fact that a function $f : X \to Y$ is Baire class one if and only if for every closed K in X, $f|_K$ has at least one point of continuity in K. In this paper, we shall prove Theorem 2 in a more general setting using Lebesgue's theorem.

2 A New Proof

Throughout the paper, (X, d_X) and (Y, d_Y) are assumed to be separable metric spaces. Denote the minimum between a and b by $a \wedge b$, the closure of a set A by $\overline{A}$ and its diameter by diam $\{A\}$. Also denote the open ball with center $x_0 \in X$ and radius $\delta > 0$ by $N_\delta(x_0)$, that is, $N_\delta(x_0) = \{y \in X : d_X(x_0, y) < \delta\}$.

Before we provide the proof of our main theorem, let us recall first the following important results. Proposition 3 is stated for the real number line in [8, Lemma 1] without proof. On the other hand, Lemma 4 is proved in [8, Lemma 2] for the real line. Though Proposition 3 is quite well-known it is hard to find a proof for general spaces in the literature. For the sake of completeness, we shall give proofs of Proposition 3 and Lemma 4 in space X by adapting the proofs found in [3, p. 75] and [8, Lemma 2], respectively.

Proposition 3. *If $X = \bigcup_{n=1}^{+\infty} E_n$ with each E_n an F_σ set in X, then there are disjoint F_σ sets F_n , $n = 1, 2, \dots$, in X such that $F_n \subseteq E_n$ and $X = \bigcup_{n=1}^{+\infty} F_n$.*

Proof. For every n, $E_n = \bigcup_{i=1}^{+\infty} E_n^i$ for some sequence $\{E_n^i\}$ of closed sets in X. Thus, we can express $X = \bigcup_{k=1}^{+\infty} A_k$ where for each k, A_k is closed in X and $A_k \subseteq E_n$ for some n. Let $H_1 = A_1$ and $H_k = A_k - (A_1 \cup A_2 \cup \cdots \cup A_{k-1})$ for $k \geq 2$. Notice that each H_k is F_σ in X, $X = \bigcup_{k=1}^{+\infty} H_k$ and $H_i \cap H_j = \emptyset$ for $i \neq j$. For each i, let $N_i = \{k \in \mathbb{N} : H_k \subseteq E_i\}$. The sets $F_i = \bigcup_{k \in N_i} H_k$ are pairwise disjoint F_σ sets, $F_i \subseteq E_i$ for each i and $X = \bigcup_{i=1}^{+\infty} F_i$. □

Lemma 4. *Let $X = \bigcup_{n=1}^{+\infty} F_n$ where F_n's are disjoint F_σ sets. Then there is a positive function $\delta(\cdot)$ on X such that $x \in F_n$, $y \in F_m$ and $n \neq m$ imply*

$$d_X(x, y) \geq \delta(x) \wedge \delta(y).$$

Proof. Since each F_n is F_σ set, then there exists a sequence $\{F_n^i\}_{i=1}^{+\infty}$ of closed sets such that $F_n = \bigcup_{i=1}^{+\infty} F_n^i$ and $F_n^i \subseteq F_n^{i+1}$ for all i. For each n and $x \in F_n$, let $x \in F_n^{i_x}$, but $x \notin F_n^j$ for all $j < i_x$. Define a function $\delta : X \to \mathbb{R}^+$ such that

$$F_m^j \cap N_{\delta(x)}(x) = \emptyset$$

for all $m \neq n$ and $m + j \leq n + i_x$. Let $x \in F_n$ and $y \in F_m$, $m \neq n$. If $n + i_x \leq m + i_y$, then $F_n^{i_x} \cap N_{\delta(y)}(y) = \emptyset$. Hence, $d_X(x, y) \geq \delta(y)$. On the other hand, if $m + i_y \leq n + i_x$, then $F_m^{i_y} \cap N_{\delta(x)}(x) = \emptyset$. Thus, $d_X(x, y) \geq \delta(x)$. All these show that $d_X(x, y) \geq \delta(x) \wedge \delta(y)$. The lemma follows. □

We are now ready to state and prove our main theorem. Recall that X and Y are assumed only to be separable metric spaces. Hence, our result generalizes the theorem of P.Y. Lee, W.K. Tang and D. Zhao [6].

Theorem 5. *Let X and Y be separable metric spaces. The following statements are equivalent.*

(1) $f : X \to Y$ is Baire class one

(2) *For each $\epsilon > 0$ there is a positive function $\delta : X \to \mathbb{R}^+$ such that for any $x, y \in X$*

$$d_X(x,y) < \delta(x) \wedge \delta(y) \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

Proof. (1) $\Rightarrow$ (2). Suppose $f : X \to Y$ is Baire class one. Let $\epsilon > 0$ be given. Find a natural number k such that $\frac{1}{k} < \epsilon$. By Lebesgue's theorem, there exists a sequence of closed sets $\{E_n\}_{n=1}^{+\infty}$ in X such that $X = \bigcup_{n=1}^{+\infty} E_n$ and $\omega_f(E_n) < \frac{1}{k}$ for each n. There exists a sequence of F_σ sets $\{F_n\}_{n=1}^{+\infty}$ in X such that $X = \bigcup_{n=1}^{+\infty} F_n$, $F_n \subseteq E_n$ for each n and $F_i \cap F_j = \emptyset$ for $i \neq j$. By Lemma 4, there is a positive function $\delta : X \to \mathbb{R}^+$ such that if $x \in F_m$ and $y \in F_n$ with $m \neq n$ implies

$$d_X(x,y) \geq \delta(x) \wedge \delta(y).$$

Let $x, y \in X$ and $d_X(x,y) < \delta(x) \wedge \delta(y)$. By the property of δ there is a unique n such that $x, y \in F_n$. Since $F_n \subseteq E_n$ and $\omega_f(E_n) < \frac{1}{k}$ it immediately follows that $d_Y(f(x), f(y)) < \frac{1}{k} < \epsilon$.

(2) $\Rightarrow$ (1) This direction is proved using the ideas from [7].
Suppose for each $\epsilon > 0$ there is a positive function $\delta : X \to \mathbb{R}^+$ such that for any $x, y \in X$

$$d_X(x,y) < \delta(x) \wedge \delta(y) \Longrightarrow d_Y(f(x), f(y)) < \frac{\epsilon}{3}.$$

For each n, let $A_n = \left\{x \in X : \delta(x) > \frac{1}{n}\right\}$ and find a closed cover $\left\{F_n^k\right\}_{k=1}^{+\infty}$ of X such that diam$\left\{F_n^k\right\} < \frac{1}{n}$ for each k. Notice that $\left\{F_n^k\right\}_{k=1}^{+\infty}$ exists because X is a separable metric space. Hence, we can write the space X as

$$X = \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{+\infty} \left(F_n^k \cap \overline{A_n}\right).$$

Let $n, k \in \mathbb{N}$ and $x, y \in F_n^k \cap \overline{A_n}$. Since $\{A_n\}$ is an increasing sequence of sets then there is a $j > n$ such that $x, y \in A_j$. We can find $x_1, y_1 \in A_n$ such that

$$0 \leq d_X(x, x_1) < \frac{1}{j}\ ,\ 0 \leq d_X(y, y_1) < \frac{1}{j} \ \text{ and } \ d_X(x_1, y_1) < \frac{1}{n}.$$

Since $A_n \subseteq A_j$, then $x_1, y_1 \in A_j$. Consequently,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_1)) + d_Y(f(x_1), f(y_1)) + d_Y(f(y), f(y_1)) < \epsilon.$$

This implies the Lebesgue's theorem. The proof is complete. □

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