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Characterization of completely $k$-magic regular graphs

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Abstract. Let $k \in \mathbb{N}$ and $c \in \mathbb{Z}_k$. A graph $G$ is said to be $c$-sum $k$-magic if there is a labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$ such that $\sum_{uv \in E(G)} \ell(uv) \equiv c \pmod{k}$ for every vertex $v$ of $G$, where $N(v)$ is the neighborhood of $v$ in $G$. We say that $G$ is completely $k$-magic whenever it is $c$-sum $k$-magic for every $c \in \mathbb{Z}_k$. In this paper, we characterize all completely $k$-magic regular graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple (unless otherwise stated) graph with vertex set $V(G)$ and edge set $E(G)$. A factor of $G$ is a subgraph $H$ with $V(H) = V(G)$. In particular, if a factor $H$ of $G$ is $h$-regular, then we say that $H$ is an $h$-factor of $G$. An $h$-factorization of $G$ is a partition of $E(G)$ into disjoint $h$-factors. If such factorization of $G$ exists, then we say that $G$ is $h$-factorable.

The following theorem is attributed to Petersen [7], which we state using the versions of Akiyama and Kano [2] and Wang and Hu [10].

**Theorem 1.1** ([2, Theorem 3.1], [7], [10, Theorem 10]). Let $G$ be a $2r$-regular connected general graph (not necessarily simple), where $r \geq 1$. Then $G$ is 2-factorable, and it has a $2k$-factor for every $k$, $1 \leq k \leq r$. Moreover, if $G$ is of even order, then it is $r$-factorable.

A graph $G$ is $\lambda$-edge connected if it remains connected whenever fewer than $\lambda$ edges are removed.

**Theorem 1.2.** [6] Let $r$ and $k$ be integers such that $1 \leq k < r$, and $G$ be a $\lambda$-edge connected $r$-regular general graph, where $\lambda \geq 1$. If one of the following conditions holds:

1. $r$ is even, $k$ is odd, $|G|$ is even, and $\frac{r}{k} \leq k \leq r(1 - \frac{1}{k})$;
2. $r$ is odd, $k$ is even, and $2 \leq k \leq r(1 - \frac{1}{k})$, or
3. $r$ and $k$ are both odd and $\frac{r}{k} \leq k$,

then $G$ has a $k$-regular factor.

Let $k$ be a positive integer. A finite simple graph $G = (V(G), E(G))$ is said to be $k$-magic if there exists an edge labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$, where $\mathbb{Z}_1 = \mathbb{Z}$ the group of integers, and $\mathbb{Z}_k = \{0, 1, 2, \ldots, k - 1\}$ the group of integers modulo $k \geq 2$, such that the induced vertex labeling $\ell^+: V(G) \to \mathbb{Z}_k$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, is a constant map. If $c \in \mathbb{Z}_k$
and $\ell^+(v) = c$ for all $v \in V(G)$, then we call $c$ is a magic sum of $G$. In particular, if $G$ is $k$-magic with magic sum $c$, then we say that $G$ is $c$-sum $k$-magic. If $G$ is $c$-sum $k$-magic for all $c \in \mathbb{Z}_k$, then it is said to be completely $k$-magic. The set of all magic sums $c \in \mathbb{Z}_k$ of $G$ is the sum spectrum of $G$ with respect to $k$ and is denoted by $\Sigma_k(G)$. If $c = 0$, then we say that $G$ is zero-sum k-magic. The null set of $G$, denoted by $N(G)$, is the set of all positive integers $k$ such that $G$ is a zero-sum $k$-magic graph.

**Remark 1.3.** If $c \in \mathbb{Z}_k$ and $\ell$ is a $c$-sum $k$-magic labeling of $G$, then the labeling $\ell'$, defined by $\ell'(e) = k - \ell(e)$, is a $(k-c)$-sum $k$-magic labeling of $G$.

**Remark 1.4.** Any 2-magic graph is not completely 2-magic.


Using the term “index set,” Wang and Hu [10] initially studied the concept of completely $k$-magic graphs. They gave a partial list of completely 1-magic regular graphs. Eniego and Garces [5] completely added the remaining cases in this list. They also presented the sum spectra of some regular graphs that are not completely $k$-magic.

**Theorem 1.5** ([1, Theorem 13]). Let $G$ be an $r$-regular graph, where $r \geq 3$ and $r \neq 5$. If $r$ is even, then $N(G) = \mathbb{N}$ (the set of positive integers); otherwise, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

**Theorem 1.6** ([4, Theorem 2.1]). Every 5-regular graph admits a zero-sum 3-magic labeling.

**Theorem 1.7** ([5, Theorem 3.3]). Let $n \geq 3$ and $k \geq 3$ be integers, and $C_n$ the cycle with $n$ vertices.

1. If $n$ is even, then $C_n$ is completely $k$-magic for all $k$.
2. If $n$ is odd, then $C_n$ is not completely $k$-magic for any $k$. Moreover, we have

$$
\Sigma_k(C_n) = \begin{cases} 
\mathbb{Z}_k \setminus \{0\} & \text{if } k \text{ is odd}, \\
\{0, 2, \ldots, k - 2\} & \text{if } k \text{ is even}.
\end{cases}
$$

**Theorem 1.8** ([5, Lemma 3.4]). Let $k \geq 4$ be an even integer. Then there exists no $k$-magic graph of odd order that is completely $k$-magic. In particular, if $c$ is a magic sum of a $k$-magic graph of odd order, then $c$ must be even.

**Theorem 1.9** ([5, Theorem 3.6]). Let $k, r \geq 3$ be integers, and $G$ an $r$-regular graph. If $\gcd(r, k) = 1$, then $\{1, 2, \ldots, k - 1\} \subseteq \Sigma_k(G)$.

**Theorem 1.10** ([5, Theorem 3.7]). Let $G$ be a zero-sum $k$-magic $r$-regular graph, where $k \geq 3$ and $r \geq 3$. If $G$ has a 1-factor, then $G$ is completely $k$-magic.

**Theorem 1.11** ([10, Theorem 13], [5, Theorem 2.1]). Let $G$ be an $r$-regular graph of order $n$. Then

$$
\Sigma_1(G) = \begin{cases} 
\mathbb{Z} \setminus \{0\} & \text{if } r = 1, \\
\mathbb{Z} & \text{if } r = 2 \text{ and } G \text{ contains even cycles only}, \\
2\mathbb{Z} \setminus \{0\} & \text{if } r = 2 \text{ and } G \text{ contains an odd cycle}, \\
2\mathbb{Z} & \text{if } r \geq 3, r \text{ even, and } n \text{ even}, \\
\mathbb{Z} & \text{if } r \geq 3 \text{ and } n \text{ even},
\end{cases}
$$

where $2\mathbb{Z}$ is the set of all even integers.
With Remark 1.4 and Theorem 1.11, it remains to characterize all completely $k$-magic regular graphs for $k \geq 3$. This characterization is the main theorem of this paper, which we state as follows.

**Theorem 1.12 (Main Theorem).** Let $r \geq 2$ and $k \geq 3$ be integers, and $G$ an $r$-regular graph of order $n \geq 3$. Then $G$ is completely $k$-magic if and only if one of the following properties holds:

1. $k \geq 3$, $r = 2$, and $G$ contains even cycles only,
2. $k \geq 5$ and $r \geq 3$ odd,
3. $k \geq 5$, $r \geq 4$ even, and $n$ even,
4. $k \geq 5$ odd, $r \geq 4$ even, and $n$ odd,
5. $k = 4$, $r \geq 3$, $n$ even, and $G$ zero-sum 4-magic, or
6. $k = 3$ and any one of the following conditions holds:
   - $r \equiv 0 \pmod 3$,
   - $r \equiv 0 \pmod 6$,
   - $r \equiv 0 \pmod 3$, $r$ odd, and $G$ has a factor $H$ such that $d_H(v) \equiv 1 \pmod 3$ for all $v \in V(H)$.

For convenience, we only consider graphs that are finite and simple (unless otherwise stated). We also write $\mathbb{Z}_k^*$ to mean $\mathbb{Z}_k \setminus \{0\}$. For graph-theoretic terms that are not explicitly defined in this paper, see [3].

### 2. Proof of the Main Theorem

We divide the proof into several results.

It is not difficult to see that if $G$ is 1-regular, then $\Sigma_k(G) = \mathbb{Z}_k^*$. For 2-regular graphs, the following remark is a consequence of Theorem 1.7.

**Remark 2.1.** Let $k \geq 3$ and $G$ a 2-regular graph. If $G$ has an odd cycle, then

$$
\Sigma_k(G) = \begin{cases} 
\mathbb{Z}_k^* & \text{if } k \text{ is odd} \\
\{0, 2, \ldots, k - 2\} & \text{if } k \text{ is even}.
\end{cases}
$$

Otherwise, we have $\Sigma_k(G) = \mathbb{Z}_k$.

Clearly, if $G$ is 1-factorable, then $G$ is completely $k$-magic. The following theorem considers regular graphs that has a factor that is completely $k$-magic.

**Theorem 2.2.** Let $r \geq 2$, $2 \leq h \leq r$, $k \neq 2$, and $G$ an $r$-regular graph. If $G$ has an $h$-factor that is completely $k$-magic, then $G$ is completely $k$-magic.

**Proof.** The case when $h = r$ is trivial, so we assume $h < r$. Let $H$ be an $h$-factor of $G$ that is completely $k$-magic. Let $\alpha = c - (r - h) \pmod k$ and $f_\alpha$ be an $\alpha$-sum $k$-magic labeling of $H$ for each $c \in \mathbb{Z}_k$.

Define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$
\ell_c(e) = \begin{cases} 
f_\alpha(e) & \text{if } e \in E(H) \\
1 & \text{if } e \in E(G \setminus H).
\end{cases}
$$

Observe that $\ell_c$ is a $c$-sum $k$-magic labeling of $G$ for each $c \in \mathbb{Z}_k$. Hence, $G$ is completely $k$-magic. \qed
The following construction will be useful.

**Remark 2.3.** Let $G$ be an $r$-regular graph with $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $r \geq 1$. Then we can construct a graph $G'$ (with parallel edges) such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{e'_1, e'_2, \ldots, e'_m\}$, where $e'_i$ is a duplicate edge of $e_i$ in $G$ for each $i$ (that is, edges $e_i$ and $e'_i$ have the same end vertices). By Theorem 1.1, $G'$ has a 2-factor $H'$ for each $h$, $1 \leq h \leq r$. Also, $G' \setminus H'$ is a $(2r - 2h)$-factor of $G'$ obtained by removing the edges of $H'$ from $G'$.

**Theorem 2.4.** Let $G$ be a 5-regular graph. Then $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

**Proof.** We know from Theorem 1.11 and Theorem 1.6 that $1, 3 \in N(G)$. For $k \geq 5$, we consider two cases.

**Case 1.** Suppose $k \geq 5$ and $k \neq 8$. Using the construction described in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 2-factor and 8-factor of $G'$, respectively.

Define a zero-sum $k$-magic labeling $\ell'$ on $G'$ by

$$\ell'(e) = \begin{cases} k - 4 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Note that the labeling $\ell$ on $G$ defined by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $e_i \in E(G)$ is a zero-sum $k$-magic labeling on $G$.

**Case 2.** Suppose $k = 8$. Using again the construction in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 4-factor and 6-factor of $G'$, respectively.

Define a zero-sum labeling $\ell'$ on $G'$ by

$$\ell'(e) = \begin{cases} 2 & \text{if } e \in E(H') \\ 4 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the labeling $\ell$ on $G$ defined by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e'_i)]$ for $e_i \in E(G)$ is a zero-sum 8-magic labeling on $G$.

Therefore, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. \hfill \Box

Note that an odd-regular graph may not be zero-sum 4-magic. It was remarked in [1, Remark 10] that an odd-regular graph $G$ is not zero-sum 4-magic if $G$ has a vertex such that every edge incident to it is a cut-edge.

**Theorem 2.5.** Let $G$ be an $r$-regular graph, where $r \geq 3$ is odd and $k \geq 5$. Then $G$ is completely $k$-magic.

**Proof.** We know from Theorems 1.5 and 2.4 that $0 \in \Sigma_k(G)$. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. As constructed in Remark 2.3, let $H'$ and $G' \setminus H'$ be a 2-factor and $(2r - 2)$-factor of $G'$, respectively. We consider two cases.

**Case 1.** Suppose $r \equiv 1 \pmod{k}$. Then $\gcd(r, k) = 1$. By Theorem 1.9, $G$ is completely $k$-magic.

**Case 2.** Suppose $r \not\equiv 1 \pmod{k}$. Assume $\gcd(r, k) = d$ so that $r = ad$ and $k = bd$ for some positive integers $a$ and $b$. Note that, since $r$ is odd, $d$ is also odd. We consider two sub-cases.

**Sub-Case 2.1.** Suppose $k \geq 5$ is odd. Then $b$ is odd.

For each $e \in \mathbb{Z}_k^+ \setminus \{k - b, k - 2b\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} x & \text{if } e \in E(H') \\ \frac{1}{2}(k + b) & \text{if } e \in E(G' \setminus H'). \end{cases}$$
where $x = \frac{1}{2}(b + c)$ if $c$ is odd, and $x = \frac{1}{2}(b + c + k)$ if $c$ is even. Observe that $\ell'_c$ is a c-sum $k$-magic labeling of $G'$ for each $c \neq 0$.

For each $c \notin \{0, k - h, k - 2b\}$, define $\ell_c : E(G) \to \mathbb{Z}_k^+$ by $\ell_c(e_i) = \ell'_c(e_i) + \ell'_c(e'_i)$ for $1 \leq i \leq m$. Since $\ell'_c$ is a c-sum $k$-magic labeling of $G'$, $\ell_c$ is a c-sum $k$-magic labeling of $G$ for each $c \in \mathbb{Z}_k^+ \setminus \{k - b, k - 2b\}$.

If $k \neq 3b$, then, by Remark 1.3, $k - b, k - 2b \in \Sigma_k(G)$. If $k = 3b$, it is enough to show that $k - 2b \in \Sigma_k(G)$. To do that, we provide a different labeling using a different set of factors of $G'$.

Let $J'$ and $G' \setminus J'$ be a 4-factor and $(2r - 4)$-factor of $G'$ respectively. In addition, we let $J' = J'_1 \cup J'_2$, where $J'_1$ and $J'_2$ are 2-factors of $J'$.

Define $\ell' : E(G') \to \mathbb{Z}_k^+$ by

$$
\ell'(e) = \begin{cases} 
\frac{1}{2}(b + 1) & \text{if } e \in E(J'_1) \\
\frac{1}{2}(b - 1) & \text{if } e \in E(J'_2) \\
b & \text{if } e \in E(G' \setminus J').
\end{cases}
$$

Since $k = 3b$, $d = 3$ and $r = 3a$. Thus, the magic sum in $G'$ is given by $2[\frac{1}{2}(b + 1)] + 2[\frac{1}{2}(b - 1)] + b(2r - 4) \equiv -2b \pmod{k}$. Define $\ell : E(G) \to \mathbb{Z}_k^+$ by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $1 \leq i \leq m$. Note that $\ell$ is also a $(k - 2b)$-sum $k$-magic labeling of $G$.

**Sub-Case 2.2.** Suppose $k \geq 6$ is even. Then $b$ is even.

By labeling all the edges of $G$ with $\frac{1}{2}k$, we see that $\frac{1}{2}k \in \Sigma_k(G)$.

Suppose $r - 1 \equiv \frac{1}{2}k \pmod{k}$. For each $c \in \mathbb{Z}_k^+ \setminus \{k - 1, \frac{1}{2}k\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^+$ by

$$
\ell'_c(e) = \begin{cases} 
c & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
$$

Observe that the sum of the labels of the edges incident to each vertex in $G'$ is $2(r - 1) + 2c \equiv 2c \pmod{k}$. Using a similar argument as in Sub-Case 2.1, it can be shown that $G$ is also c-sum $k$-magic for all even $c \neq 0$. Thus, we are left to show that $G$ is c-sum $k$-magic as well for all odd $c$.

For each odd $c \neq k - 1$, define $\ell_c : E(G) \to \mathbb{Z}_k^+$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for each $i$, $1 \leq i \leq m$. Note that, since $\ell'_c$ is a 2c-sum $k$-magic labeling of $G'$, $\ell_c$ is a c-sum $k$-magic labeling of $G$ for each odd $c \neq k - 1$. Again, by Remark 1.3, we see that $k - 1 \notin \Sigma_k(G)$.

Suppose $r - 1 \equiv r_0 \pmod{k}$, where $r_0 \neq \frac{1}{2}k$. For each $c \in \mathbb{Z}_k^+ \setminus \{r_0, r_0 + \frac{1}{2}k, r_0 - 1\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^+$ by

$$
\ell'_c(e) = \begin{cases} 
c - r_0 & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
$$

Observe that the sum of the labels of the edges incident to each vertex in $G'$ is $2r_0 + 2c - 2r_0 \equiv 2c \pmod{k}$. As in Sub-Case 2.1, it can be shown that $G$ is also even-sum $k$-magic. So again, we are left to show that $G$ is odd-sum k-magic.

As what we did earlier, for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$), define $\ell_c : E(G) \to \mathbb{Z}_k^+$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for all $i$, $1 \leq i \leq m$. Since $\ell'_c$ is a 2c-sum $k$-magic labeling of $G'$, $\ell_c$ is a c-sum $k$-magic labeling of $G$ for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$). If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are not inverses, then, by Remark 1.3, $\mathbb{Z}_k^+ \subset \Sigma_k(G)$.

If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are inverses, then it is enough to show that $r_0 - 1 \in \Sigma_k(G)$. Define $\ell'$ on $G'$ by

$$
\ell'(e) = \begin{cases} 
k - 1 & \text{if } e \in E(H') \\
1 & \text{if } e \in E(G' \setminus H').
\end{cases}
$$
Note that the magic sum using $\ell'$ is $2r_0 - 2$. Define $\ell$ on $G$ by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e_i')]$ for $e_i \in E(G)$. Clearly, $\ell$ is an $(r_0 - 1)$-sum $k$-magic labeling on $G$. Thus, by Remark 1.3, $r_0 + \frac{1}{2}k \in \Sigma_k(G)$, and so $Z_k^* \subseteq \Sigma_k(G)$.

In any case, $G$ is completely $k$-magic. \hfill \Box

**Theorem 2.6.** Let $k \geq 5$ and $G$ a $2r$-regular graph of order $n \geq 3$, where $r \geq 2$.

(1) If $n$ is even, then $G$ is completely $k$-magic.

(2) If $n$ is odd, then

   (i) $G$ is completely $k$-magic if $k$ is odd, and

   (ii) $\Sigma_k(G) = \{0, 2, 4, \ldots, k - 2\}$ if $k$ is even.

**Proof.** Let $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. By Theorem 1.5, $G$ is zero-sum $k$-magic.

(1) Suppose $r = 2$. To prove the theorem, we only show that $Z_k^* \subseteq \Sigma_k(G)$. We consider two cases.

CASE 1. Suppose $k$ is odd. Then $\gcd(4, k) = 1$. By Theorem 1.9, $Z_k^* \subseteq \Sigma_k(G)$.

CASE 2. Suppose $k$ is even. It is not difficult to see that, being 4-regular, $G$ is 2-edge connected. By Remark 2.3, we can construct $G'$ so that $G'$ is a 4-edge-connected 8-regular graph. By Theorem 1.2, $G'$ has a 3-factor, say $H'$. Let $G' \setminus H'$ be the 5-factor of $G'$ obtained by removing the edges of $H'$ from $G'$.

SUB-CASE 2.1. Let $k = 2d$, $d$ even. For each $c = Z_k^* \setminus \{\frac{1}{2}k, \frac{1}{2}k\}$, define $f_c : E(G') \to Z_k^*$ by

$$f_c(e) = \begin{cases} 2c & \text{if } e \in E(H') \\ k - c & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each of the vertices in $G'$ is equal to $5(k - c) + 3(2c) \equiv c \pmod{k}$. This shows that $f_c$ is a $c$-sum $k$-magic labeling of $G'$ for all $c \neq 0, \frac{1}{2}k, \frac{1}{2}k$. By Remark 1.3, $\frac{1}{2}k \in \Sigma_k(G')$.

For each $c = Z_k^* \setminus \{\frac{1}{2}k, \frac{1}{2}k\}$, define $\ell_c : E(G) \to Z_k^*$ by $\ell_c(e_i) = f_c(e_i) + f_c(e_i')$ for all $i, 1 \leq i \leq m$. Clearly, $\ell_c$ is a $c$-sum $k$-magic labeling of $G$ for each $c = Z_k^* \setminus \{\frac{1}{2}k, \frac{1}{2}k\}$. By Remark 1.3, we see that $\Sigma_k^* \setminus \{\frac{1}{2}k\} \subseteq \Sigma_k(G)$.

By Theorem 1.1, $G$ is 2-factorable. Let $G_1$ and $G_2$ be the two 2-factors of $G$. Label the edges in $G_1$ with $d$ and the edges in $G_2$ with $\frac{1}{2}(k - d)$. This shows that $d = \frac{1}{2}k \in \Sigma_k(G)$.

SUB-CASE 2.2. Let $k = 2d$, $d \geq 3$ odd. Observe that, for $c \neq 0, \frac{1}{2}k$, the labeling $\ell_c$ in Sub-case 2.1 is a $c$-sum $k$-magic labeling of $G$. We are left to show that $\frac{1}{2}k \in \Sigma_k(G)$.

Let $d \neq 3$ and 9. We give a labeling for the factors of $G'$ defined above (namely, $H'$ and $G' \setminus H'$) and the 2-factors of $G$ (namely, $G_1$ and $G_2$) to show that $G$ is $d$-sum $k$-magic.

Let $f : E(G) \to Z_k^*$ be defined by

$$f(e) = \begin{cases} d + 1 & \text{if } e \in E(G_1) \\ \frac{1}{2}(k - d - 1) & \text{if } e \in E(G_2). \end{cases}$$

Clearly, $f$ is $(d + 1)$-sum $k$-magic labeling of $G$.

Let $g : E(G') \to Z_k^*$ be defined by

$$g(e) = \begin{cases} k - 2 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Define also $g : E(G) \to Z_k^*$ by $g(e_i) = g(e_i) + g(e'_i)$ for all $i, 1 \leq i \leq m$. Note that $g'$ is a $(k - 1)$-sum $k$-magic labeling of $G'$, so $g$ is a $(k - 1)$-sum $k$-magic labeling of $G$.
Finally, define $\ell : E(G) \to \mathbb{Z}_k^*$ by $\ell(e) = f(e) + g(e)$ for all $e \in E(G)$. Since $f$ and $g$ are $(d + 1)$-sum and $(k - 1)$-sum $k$-magic labeling of $G$, respectively, $\ell$ is a $d$-sum $k$-magic labeling of $G$.

Suppose $d = 3$ or 9. Define $g' : E(G') \to \mathbb{Z}_k^*$ be defined by

$$g'(e) = \begin{cases} 2x & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H') \end{cases},$$

where $x = 1$ if $d = 3$, and $x = 3$ if $d = 9$. Note that $g'$ is a 5-sum $k$-magic labeling of $G'$. Define a labeling $g$ on $G$ by $g(e_i) = g'(e_i) + g'(e_i') + 1$ for all $i, 1 \leq i \leq m$. Note that $g$ is a $d$-sum $k$-magic labeling on $G$. Thus, $d = \frac{1}{2}k \in \Sigma_k(G)$, and so $G$ is completely $k$-magic.

Suppose $r \geq 3$ is odd. By Theorem 1.1, $G$ is $r$-factorable. By Theorem 2.5, the $r$-factors of $G$ are completely $k$-magic for all $k \geq 5$. Thus, by Theorem 2.2, $G$ is also completely $k$-magic.

If $r \geq 4$ is even, then, by Theorem 1.1, $G$ has a 6-factor, say $H$. Using the case for $r$ is odd, $H$ is completely $k$-magic. Thus, by Theorem 2.2, $G$ is also completely $k$-magic.

(2(i)) By Theorem 1.1, $G$ is 2-factorable. Let $G_1, G_2, \ldots, G_r$ be the 2-factors of $G$. If $k$ is odd, then, by Remark 2.1, $\mathbb{Z}_k^* \subseteq \sum_k(G_i)$ for all $i, 1 \leq i \leq r$. For each $i$ and $c \in \mathbb{Z}_k^*$, let $\ell_i^c$ be a $c$-sum $k$-magic labeling of $G_i$. We consider two cases.

Case 1. Suppose $r \equiv 1 \pmod{k}$. For each $c \in \mathbb{Z}_k^*$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} \ell_1^c(e) & \text{if } e \in E(G_1) \\ \ell_i^c(e) & \text{if } e \in E(G_i) \text{ for some } i \neq 1 \end{cases}$$

Note that $\ell_c$ is a $c$-sum $k$-magic labeling of $G$ for all $c \neq 0$.

Case 2. Suppose $r \equiv 1 \pmod{k}$. For each $c \in \mathbb{Z}_k^* \setminus \{r - 1 \pmod{k} \}$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} \ell_{x}^c(e) & \text{if } e \in E(G_1) \\ \ell_i^c(e) & \text{if } e \in E(G_i) \text{ for some } i \neq 1 \end{cases}$$

where $x \equiv r - 1 \pmod{k}$. The sum of the labels of the edges incident to each vertex is $c$ (mod $k$). Thus, $G$ is $c$-sum $k$-magic for each $c \neq x$. By Remark 1.3, $G$ is $x$-sum $k$-magic since $G$ is $(k - x)$-sum $k$-magic. In this case, $G$ is completely $k$-magic.

(2(ii)) This follows from Remark 2.1, Lemma 1.8, and Theorem 2.2. \qed

The proof of the following theorems are similar to Theorem 2.5 and Theorem 2.6.

**Theorem 2.7.** Let $r \geq 3$, and $G$ a zero-sum 4-magic $r$-regular graph. Then

1. If the order of $G$ is even, then $G$ is completely 4-magic.
2. If the order of $G$ is odd, then $\Sigma_4(G) = \{0, 2\}$.

**Theorem 2.8.** Let $G$ be an $r$-regular graph, where $r \geq 3$.

1. If $r \equiv 0 \pmod{3}$ or $r \equiv 0 \pmod{6}$, then $G$ is completely 3-magic.
2. If $r \equiv 0 \pmod{3}$ and $r$ odd, then $G$ is completely 3-magic if and only if $G$ has a factor $H$ such that $d_H(v) \equiv 1 \pmod{3}$ for all $v \in V(H)$. 


References


