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Arnold A. Eniego

Ian June L. Garces

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# On Completely $k$-Magic Regular Graphs

**Arnold A. Eniego and I.J.L. Garces**

Department of Mathematics
Ateneo de Manila University
1108 Loyola Heights, Quezon City
The Philippines



**Abstract**

Let $k$ be a positive integer. A graph $G = (V(G), E(G))$ is said to be $k$-magic if there is a function (or edge labeling) $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$, where $\mathbb{Z}_1 = \mathbb{Z}$, such that the induced function (or vertex labeling) $\ell^+ : V(G) \to \mathbb{Z}_k$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, is a constant map, where the sum is taken in $\mathbb{Z}_k$. We say that $G$ is $c$-sum $k$-magic if $\ell^+(v) = c$ for all $v \in V(G)$. The set of all $c \in \mathbb{Z}_k$ such that $G$ is $c$-sum $k$-magic is called the sum spectrum of $G$ with respect to $k$. In the case when the sum spectrum of $G$ is $\mathbb{Z}_k$, we say that $G$ is completely $k$-magic. In this paper, we determine all completely 1-magic regular graphs. After observing that any 2-magic graph is not completely 2-magic, we show that some regular graphs are completely $k$-magic for $k \geq 3$, and determine the sum spectra of some regular graphs that are not completely $k$-magic.

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**Keywords:** $k$-magic graphs, completely $k$-magic graphs, Hamiltonian decomposition, $h$-factorable

# 1 Introduction

Let $G$ be a finite simple graph with vertex set $V(G) \neq \varnothing$ and edge set $E(G)$, and $A$ a non-trivial Abelian group written additively. We say that $G$ is $A$-*magic* if there is a function (or edge labeling) $\ell : E(G) \to A \setminus \{0\}$ such that the induced function (or vertex labeling) $\ell^+ : V(G) \to A$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, where the sum is performed in $A$, is constant; that is, there exists a $c \in A$ for which $\ell^+(v) = c$ for all $v \in V(G)$. In such case, the element $c$ of $A$ is called a *magic sum* of $G$. A graph is *fully magic* if it is $A$-magic for any Abelian group $A$, and is *non-magic* if it is not $A$-magic for any Abelian group $A$.

The concept of $A$-magic graph was introduced by Sedlacek [12]. He defined $A$-magic graphs as a graph with real-valued edge labeling such that distinct edges have distinct nonnegative labels, and the sum of the labels of the edges incident to any vertex is constant. For the literature and development on the concept of $A$-magic graphs, readers are referred to [1], [10], [11], and [12].

Let $\mathbb{Z}$ be the Abelian group of integers, and $\mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$ be the Abelian group of integers modulo $k \geq 2$. If $G$ is a $\mathbb{Z}_k$-magic graph, then we say that $G$ is *$k$-magic*. For convenience, we let $\mathbb{Z}_1 = \mathbb{Z}$, and $\mathbb{Z}$-magic graph will be considered as 1-magic. In particular, if $G$ is $k$-magic with magic sum $c$, then we say that $G$ is *$c$-sum $k$-magic*. If $c = 0$, then we say that $G$ is a *zero-sum $k$-magic* graph. The *null set* of $G$, denoted by $N(G)$, is the set of all positive integers $k$ such that $G$ is a zero-sum $k$-magic graph.

**Theorem 1.1.** [11, Theorem 4.2] *Let $n \geq 3$ be an integer. Then*

$$N(C_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is even} \\ 2\mathbb{N} & \text{if } n \text{ is odd,} \end{cases}$$

*where* $\mathbb{N} = \{1, 2, 3, \ldots\}$ *and* $2\mathbb{N} = \{2, 4, 6, \ldots\}$.

A graph is said to be *$r$-regular*, or simply *regular*, if the degree of each vertex is $r$.

**Theorem 1.2.** [1, Theorem 13] *Let $G$ be an $r$-regular graph, where $r > 3$ and $r \neq 5$. If $r$ is even, then $N(G) = \mathbb{N}$; otherwise, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.*

A subset $M$ of $E(G)$ is called a *matching* in $G$ if no two edges in $M$ are adjacent in $G$; that is, no two edges in $M$ share a common vertex. A matching $M$ *saturates* a vertex $v$, and $v$ is said to be *$M$-saturated*, if some edge of $M$ is incident with $v$. A matching $M$ is said to be *perfect* if every vertex of $G$ is $M$-saturated.

A *Hamiltonian cycle* of $G$ is a cycle that contains every vertex of $G$. A graph is said to be *Hamiltonian* if it has a Hamiltonian cycle.

Let $G$ be an $r$-regular graph with edge set $E(G)$. Then $G$ is said to have *Hamiltonian decomposition* if either (1) $r = 2d$ and $E(G)$ can be partitioned into $d$ Hamiltonian cycles, or (2) $r = 2d + 1$ and $E(G)$ can be partitioned into $d$ Hamiltonian cycles and a perfect matching.

**Theorem 1.3.** *The following graphs admit Hamiltonian decomposition:*

(1) *complete graph* $K_n$, $n \geq 2$ (see [8, p. 162, p. 176]);

(2) *complete bipartite graph* $K_{n,n}$, $n \geq 1$ (see [8, p. 162, p. 176]);

(3) $n$*-cube* $Q_n$, $n \geq 1$ (see [3, Proposition 1]); *and*

(4) *generalized Petersen graph* $P_{m,n}$, $n \geq 4$, $1 \leq m < \lfloor n/2 \rfloor$.

A graph $H$ is a *subgraph* of $G$ if $\varnothing \subsetneq V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *factor* of $G$ is a subgraph $H$ with $V(H) = V(G)$. In particular, if a factor $H$ of $G$ is $h$-regular, then we say that $H$ is an *$h$-factor* of $G$. An *$h$-factorization* of $G$ is a partition of the edges of $G$ into disjoint $h$-factors. If such factorization of $G$ exists, then we say that $G$ is *$h$-factorable*.

The following theorem is generally due to Petersen, and has been strengthened by many authors.

**Theorem 1.4.** ([2, Theorem 3.1], [9], [13, Theorem 10]) *For every integer* $r \geq 1$, *every* $2r$*-regular graph is* 2*-factorable. In particular, for every integer* $k$, $1 \leq k \leq r$, *every* $2r$*-regular graph has a* $2k$*-factor. Moreover, every connected* $2r$*-regular graph of even order is* $r$*-factorable.*

A graph $G$ is $\lambda$*-edge connected* if it remains connected whenever fewer than $\lambda$ edges are removed.

**Theorem 1.5.** ([7], [2, Theorem 3.4]) *Let* $r$, $k$, *and* $\lambda$ *be integers such that* $1 \leq k < r$ *and* $\lambda \geq 1$, *and* $G$ *be a* $\lambda$*-edge-connected* $r$*-regular graph* (*that may have loops and parallel edges*). *If* $r$ *is even,* $k$ *is odd,* $|V(G)|$ *is even, and* $\frac{r}{\lambda} \leq k \leq r(1 - \frac{1}{\lambda})$, *then* $G$ *has a* $k$*-factor.*

Finally, we now present the main definition of the paper.

**Definition 1.6.** Let $G$ be a $k$-magic graph for some positive integer $k$. The set of all $c \in \mathbb{Z}_k$ such that $G$ is $c$-sum $k$-magic is called the *sum spectrum of $G$ with respect to $k$*, and is denoted by $\Sigma_k(G)$. Moreover, in the special case when $\Sigma_k(G) = \mathbb{Z}_k$, we say that $G$ is *completely $k$-magic*.

For $k = 1$, the characterization of completely 1-magic regular graphs was studied in [13]. Using the term "index set" in their paper, the authors determined the sum spectra of almost all regular graphs.

**Theorem 1.7.** [13, Theorem 13 (corrected)] *Let* $G$ *be an* $r$*-regular graph of order* $n$. *Then*

$$\Sigma_1(G) = \begin{cases} \mathbb{Z} \setminus \{0\} & \text{if } r = 1 \\ \mathbb{Z} & \text{if } r = 2 \text{ and } G \text{ contains even cycles only} \\ 2\mathbb{Z} \setminus \{0\} & \text{if } r = 2 \text{ and } G \text{ contains an odd cycle} \\ 2\mathbb{Z} & \text{if } r \geq 3,\ r \text{ even, and } n \text{ odd} \\ \mathbb{Z} & \text{if } r \geq 3,\ r \neq 4k,\ k \geq 1, \text{ and } n \text{ even,} \end{cases}$$

*where* $2\mathbb{Z}$ *is the set of all even integers.*

In this present paper, we first solve the remaining case of the sum spectrum of 1-magic regular graphs left out in [13], and determine all completely 1-magic regular graphs. After observing that any 2-magic graph is not completely 2-magic, we show that some regular graphs are completely $k$-magic for $k \geq 3$ by considering regular graphs that are hamiltonian, that has hamiltonian decomposition, or that has perfect matching. We also determine the sum spectra of some regular graphs that are not completely $k$-magic.

For convenience, we assume that all graphs to be considered are connected, and we write $\mathbb{Z}_k^*$ to mean $\mathbb{Z}_k \setminus \{0\}$. For graph-theoretic terms that are not explicitly defined in this paper, refer to [4].

# 2 Completely 1-Magic Regular Graphs

As mentioned earlier, the authors in [13] left out a case in determining the sum spectra of 1-magic regular graphs. The following theorem solves this undetermined case.

**Theorem 2.1.** *Suppose* $t \geq 1$ *is an integer, and* $G$ *is a* $4t$*-regular graph of even order. Then* $\Sigma_1(G) = \mathbb{Z}$*.*

*Proof.* We consider two cases.

Case 1. Suppose $t = 1$. It is not difficult to see that, being 4-regular, $G$ is 2-edge connected. Since $0 \in \Sigma_1(G)$ as guaranteed by Theorem 1.2, it is enough to show that $\mathbb{Z}^* \subset \Sigma_1(G)$.

Let $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. We construct a graph $G'$ (with parallel edges) such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{e'_1, e'_2, e'_3, \ldots, e'_m\}$, where $e'_i$ is a duplicate edge of $e_i$ in $G$ for each $i$ (that is, edges $e_i$ and $e'_i$ have the same end vertices). Then $G'$ is a 4-edge-connected 8-regular graph. By Theorem 1.5, $G'$ has a 5-factor, say $H$. Let $G' \setminus H$ be the 3-factor of $G'$ obtained by removing the edges of $H$ from $G'$.

For each $c \in \mathbb{Z}^*$, define $f_c : E(G') \to \mathbb{Z}^*$ by

$$f_c(e) = \begin{cases} -c & \text{if } e \in E(H) \\ 2c & \text{if } e \in E(G' \setminus H). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex of $G'$ is equal to $5(-c) + 3(2c) = c$. This shows that $f_c$ is a $c$-sum 1-magic labeling of $G'$ for every $c \in \mathbb{Z}^*$.

For each $c \in \mathbb{Z}^*$, define $\ell_c : E(G) \to \mathbb{Z}^*$ by $\ell_c(e_i) = f_c(e_i) + f_c(e'_i)$ for each $i$, $1 \leq i \leq m$. Clearly, $\ell_c$ is a $c$-sum 1-magic labeling of $G$ for each $c \in \mathbb{Z}^*$. Thus, we have $\mathbb{Z}^* \subset \Sigma_1(G)$.

Case 2. Suppose $t \geq 2$. By Theorem 1.4, $G$ has a $2y$-factor for each $1 \leq y \leq 2t$. In particular, $G$ has a 6-factor, say $H$. By Theorem 1.7, we have $\Sigma_1(H) = \mathbb{Z}$.

Let $G \setminus H$ be the $(4t-6)$-factor of $G$ obtained by removing the edges of $H$ from $G$.

For each $c \in \mathbb{Z}$, let $f_c$ be a $c$-sum 1-magic labeling of $H$. Define $\ell_c : E(G) \to \mathbb{Z}^*$ by

$$\ell_c(e) = \begin{cases} f_{c-(4t-6)}(e) & \text{if } e \in E(H) \\ 1 & \text{if } e \in E(G \setminus H). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex of $G$ is $c - (4r-6) + (4r-6) = c$. Thus, $\ell_c$ is a $c$-sum $k$-magic labeling of $G$ for each $c \in \mathbb{Z}$, and so $\Sigma_1(G) = \mathbb{Z}$. $\square$

The preceding theorem, together with Theorem 1.7, gives the following characterization of completely 1-magic regular graphs.

**Corollary 2.2.** *Let $r$ be a positive integer, and $G$ an $r$-regular graph. Then $G$ is completely 1-magic if and only if one of the following properties holds:*

(1) *$r = 2$ and $G$ contains even cycles only, or*

(2) *$r \geq 3$ and $n$ even.*

# 3 Completely $k$-Magic Regular Graphs, $k \geq 2$

We first present two special observations.

**Observation 3.1.** *Any 2-magic graph is not completely 2-magic. To see this, for a graph to be 2-magic, the degrees of the vertices must have the same parity. Since the only possible label of an edge is 1, any 2-magic graph is either zero-sum 2-magic (when the degree of each vertex is even) or 1-sum 2-magic (when the degree of each vertex is odd). Thus, we have either $\Sigma_2(G) = \{0\}$ or $\Sigma_2(G) = \{1\}$.*

**Observation 3.2.** *It is not difficult to observe that any regular graph is fully magic. In particular, any regular graph is $k$-magic for any positive integer $k$. However, fully magicness does not imply completely $k$-magicness. The path $P_2$ is a 1-regular graph, so it is $k$-magic for all $k$. However, $P_2$ is not completely $k$-magic since it is not zero-sum $k$-magic for all $k$. The sum spectrum of $P_2$ is clearly $\mathbb{Z}_k^*$ for any $k$.*

Due to Observation 3.1, it suffices to investigate regular graphs that are completely $k$-magic for $k \geq 3$.

If a connected graph is 2-regular, then it must be a cycle. Clearly, the cycle $C_n$ is not completely 2-magic for any integer $n \geq 3$, and $\Sigma_2(C_n) = \{0\}$. Our first result characterizes the complete $k$-magicness of cycles $C_n$ for $n \geq 3$ and $k \geq 3$.

**Theorem 3.3.** *Let $n \geq 3$ and $k \geq 3$ be integers.*

(1) *If $n$ is even, then $C_n$ is completely $k$-magic for all $k$.*

(2) *If $n$ is odd, then $C_n$ is not completely $k$-magic for any $k$. Moreover, we have*

$$\Sigma_k(C_n) = \begin{cases} \mathbb{Z}_k \setminus \{0\} & \text{if } k \text{ is odd,} \\ \{0, 2, \ldots, k-2\} & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* (1) By Theorem 1.1, $C_n$ is zero-sum $k$-magic graph for any $k \geq 3$.

To get a magic sum of 1, label the edges of $C_n$ alternately with 2 and $k-1$, and, to get a magic sum of $c \in \mathbb{Z}_k \setminus \{0, 1\}$, label the edges of $C_n$ alternately with 1 and $c - 1$.

(2) By Theorem 1.1, $C_n$ is zero-sum $k$-magic graph for any even $k \geq 4$.

Suppose $k$ is odd. Let $c \in \mathbb{Z}_k^*$. If $c$ is even, we label all the edges of $C_n$ with $\frac{1}{2}c$. If $c$ is odd, we label all the edges of $C_n$ with $\frac{1}{2}(k + c)$. Note that, in either case, the sum of the labels of the edges incident to each vertex is $c \pmod{k}$, which implies that $\Sigma_k(C_n) = \mathbb{Z}_k^*$.

Suppose $k$ is even. To obtain an even magic sum of $c \geq 2$, we simply label all the edges of $C_n$ with $\frac{1}{2}c$. On the other hand, to get an odd magic sum, the edges of $C_n$ must be alternately labeled with odd and even elements of $\mathbb{Z}_k^*$, which cannot be done because $n$ is odd. It follows that $\Sigma_k(C_n) = \{0, 2, \ldots, k-2\}$. □

Before continuing further, we present a lemma that specifies which $k$-magic graphs for even $k$ are not completely $k$-magic.

**Lemma 3.4.** *Let $k \geq 4$ be an even integer. Then there exists no $k$-magic graph of odd order that is completely $k$-magic. In particular, if $c$ is a magic sum of a $k$-magic graph of odd order, then $c$ must be even.*

*Proof.* Suppose, on the contrary, that there is a $k$-magic graph $G$ of odd order that is completely $k$-magic. Let $c \in \mathbb{Z}_k$ be odd, and let $\ell$ a $c$-sum $k$-magic labeling of $G$. We let $V(G) = \{u_1, u_2, \ldots, u_m\}$. Then the computation for the induced labeling $\ell^+$ is as follows:

$$\ell^+(u_i) = \sum_{u \in N_G(u_i)} \ell(uu_i) \equiv c \pmod{k}$$

for all $1 \leq i \leq m$, where $N_G(u) = \{v \in V(G) | v \text{ is adjacent to } u\}$. Adding all these congruences gives

$$\sum_{i=1}^{m} \sum_{v \in N_G(u_i)} \ell(u_i v) \equiv mc \pmod{k}.$$

In the last congruence, since the left-hand-side double-sum expression is even, while $mc$ is odd, we get a contradiction. Thus, a $k$-magic graph of odd order cannot have an odd magic sum. $\square$

The following lemma is an exercise in [6].

**Lemma 3.5.** *Let $k$ be a positive integer.*

(1) *Let $a \in \mathbb{Z}_k$. Then $\langle a \rangle = \mathbb{Z}_k$ if and only if gcd$(a, k) = 1$.*

(2) *If gcd$(a, k) = d \neq 1$, then $\langle a \rangle \subset \mathbb{Z}_k$. In particular, we have $|\langle a \rangle| = \frac{k}{d}$.*

(3) *Let $a, b \in \mathbb{Z}_k$. Then*

(i) *the equation $x + a = b$ has a unique solution in $\mathbb{Z}_k$, and the solution is given by $x \equiv b - a \pmod{k}$;*

(ii) *the equation $2x + a = b$ has a unique solution in $\mathbb{Z}_k$ if $k$ is odd; and*

(iii) *the equation $2x + a = b$ does not have a solution in $\mathbb{Z}_k$ if $k$ is even, and $a$ and $b$ have different parity.*

Let $k \geq 3$ and graph $G$ be $r$-regular, where $r \geq 3$. Labeling each edge of $G$ with 1 produces a magic sum $r \pmod{k}$. In general, by labeling the edges of $G$ with the same element $d$ produces a magic sum $rd \pmod{k}$. This implies that $\{c \in \mathbb{Z}_k | c \equiv rd \pmod{k}, d \in \mathbb{Z}_k\} \subseteq \Sigma_k(G)$. Using this labeling technique and by Lemma 3.5, we obtain the following theorem.

**Theorem 3.6.** *Let $k, r \geq 3$ be integers, and $G$ an $r$-regular graph.*

(1) *If gcd$(r, k) = 1$, then $\Sigma_k(G) = \{1, 2, \ldots, k-1\}$ if and only if $G$ is not zero-sum $k$-magic; otherwise, $G$ is completely $k$-magic.*

(2) *If gcd$(r, k) = d \neq 1$, then*

$$\{c \in \mathbb{Z}_k | c = 0 \text{ or } c \equiv bd \pmod{k}, b \in \mathbb{Z}_k\} \subseteq \Sigma_k(G).$$

**Theorem 3.7.** *Let $G$ be a zero-sum $k$-magic $r$-regular graph, where $k \geq 3$ and $r \geq 3$. If $G$ has a perfect matching, then $G$ is completely $k$-magic.*

*Proof.* Let $M$ be a perfect matching in $G$. Since $G$ is zero-sum $k$-magic, it is enough to show that $G$ is $c$-sum $k$-magic for all $c \in \mathbb{Z}_k^*$. We consider two cases:

CASE 1. Suppose $r \equiv 1 \pmod{k}$. Then $\gcd(r,k) = 1$. By Theorem 3.6, $\{1, 2, \ldots, k-1\} \subseteq \Sigma_k(G)$. Thus, $G$ is completely $k$-magic.

CASE 2. Suppose $r \not\equiv 1 \pmod{k}$. We consider two sub-cases.

SUB-CASE 2.1. Let $c \in \mathbb{Z}_k^*$ with $c \equiv r-1 \pmod{k}$. Label each edge in $M$ with $k-(r-1)$, and the edges not in $M$ with 2. Since $r-1$ is non-zero in $\mathbb{Z}_k$, $k-(r-1) \pmod{k}$ is also non-zero. Then the sum of the labels at each vertex is $2(r-1)+k-(r-1) \equiv r-1 \equiv c \pmod{k}$.

SUB-CASE 2.2. Let $c \in \mathbb{Z}_k^*$ with $c \not\equiv r-1 \pmod{k}$. Label each edge in $M$ with $c-(r-1) \pmod{k}$, and all edges not in $M$ with 1. Then the sum of the labels at each vertex is clearly $r-1+c-(r-1) \equiv c \pmod{k}$.

In any case, we see that $G$ is completely $k$-magic. □

**Theorem 3.8.** *Let $G$ be a zero-sum $k$-magic $r$-regular Hamiltonian graph, where $k \geq 3$ and $r \geq 3$.*

(1) *If the order of $G$ is even, then $G$ is completely $k$-magic.*

(2) *If the order of $G$ is odd, then*

  (i) *$G$ is completely $k$-magic if $k$ is odd, and*

  (ii) *$\Sigma_k(G) = \{0, 2, 4, \ldots, k-2\}$ if $k$ is even.*

*Proof.* We show that $G$ is $c$-sum $k$-magic for all $c \in \mathbb{Z}_k^*$.

(1) Since every cycle of even order has a perfect matching, $G$ has a perfect matching. By Theorem 3.7, $G$ is completely $k$-magic.

(2) Since the order of $G$ is odd, $r$ must be even. Let $H$ be a Hamiltonian cycle in $G$.

(i) Suppose $k \geq 3$ is odd. We consider two cases.

CASE 1. Suppose $r \equiv 2 \pmod{k}$. Let $c \in \mathbb{Z}_k^*$. Label each edge of $H$ with $x$, where $x = \frac{1}{2}c$ if $c$ is even or $x = \frac{1}{2}(k+c)$ if $c$ is odd, and label the edges in $G \setminus H$ with 1. Then the sum of the labels at each vertex of $G$ is $2x+r-2 \equiv c \pmod{k}$.

CASE 2. Suppose $r \not\equiv 2 \pmod{k}$. We consider two sub-cases.

SUB-CASE 2.1. Let $c \in \mathbb{Z}_k^* - \{x \in \mathbb{Z}_k | x \equiv r-2 \pmod{k}\}$. Label each edge of $H$ with $x$, where $x = \frac{1}{2}(c-r+2)$ if $c$ is even or $x = \frac{1}{2}(k+c-r+2)$ if $c$ is odd, and label each edge of $G \setminus H$ with 1. This labeling gives a magic sum $2x+r-2 \equiv c \pmod{k}$.

SUB-CASE 2.2. Let $c \in \mathbb{Z}_k^*$ with $c \equiv r-2 \pmod{k}$. Label the edges of $H$ with $c$, and label the edges of $G \setminus H$ with $k-1$. At each vertex, the sum of the labels of the edges incident to it is $2c+(k-1)(r-2) \equiv c \pmod{k}$.

Thus, for any $c \in \mathbb{Z}_k$, $G$ is $c$-sum $k$-magic for all odd $k$, which implies that $G$ is completely $k$-magic for odd $k$.

(ii) Suppose $k \geq 4$ is even. By Lemma 3.4, it follows that

$$\Sigma_k(G) \subseteq \{0, 2, 4, \ldots, k-2\}.$$

Now, let $c \in \{2, 4, 6, \ldots, k-2\}$. Applying the same labeling technique used in (i) above shows that $c \in \Sigma_k(G)$. Because $G$ is zero-sum $k$-magic, it follows that $\{0, 2, 4, \ldots, k-2\} \subseteq \Sigma_k(G)$.

Thus, if $k \geq 4$ is even, then $\Sigma_k(G) = \{0, 2, 4, \ldots, k-2\}$. □

**Corollary 3.9.** *Let $k \geq 3$ and $r \geq 3$. If $G$ is $r$-regular and has a Hamiltonian decomposition, then the conclusion of Theorem 3.8 holds.*

*Proof.* From the hypothesis of Theorem 3.8, we only need to establish that $G$ is a zero-sum $k$-magic graph. By Theorem 1.2, we are only left to consider the cases when $r = 5$, and when $r$ is odd and $k = 4$. We prove the more general case when $r \geq 3$ is odd.

By definition, we can write $G$ as an edge-disjoint union of a perfect matching (say, $M$) and $\frac{1}{2}(r-1)$ Hamiltonian cycles (say, $H$ as one of them). As guaranteed in Theorem 1.2, there exists a zero-sum $k$-magic labeling of $G \backslash (H \cup M)$. Together with this labeling, we label the edges of $H$ with 1 and the edges of $M$ with $k-2$, and we get a zero-sum $k$-magic labeling of $G$. □

We end this paper with several examples of completely $k$-magic graphs, whose proofs follow immediately from Corollary 3.9, together with Theorem 1.3.

**Corollary 3.10.** *Let $n \geq 4$ and $k \geq 3$. Then*

(1) *the complete graph $K_n$ is completely $k$-magic if $n$ is even, or $n$ is odd and $k$ is odd; and*

(2) $\Sigma_k(K_n) = \{0, 2, 4, \ldots, k-2\}$ *if $n$ is odd and $k$ is even.*

**Corollary 3.11.** *For $n \geq 3$ and $k \geq 3$, the regular complete bipartite graph $K_{n,n}$ is completely $k$-magic.*

**Corollary 3.12.** *For $n \geq 2$ and $k \geq 3$, the $n$-cube $Q_n$ is completely $k$-magic.*

**Corollary 3.13.** *For $n \geq 4$, $1 \leq m < \lfloor n/2 \rfloor$, and $k \geq 3$, the generalized Petersen graph $P_{m,n}$ is completely $k$-magic.*

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