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Computing the Metric Dimension of Truncated Wheels

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Abstract

For an ordered subset \( W = \{w_1, w_2, w_3, \ldots, w_k\} \) of vertices in a connected graph \( G \) and a vertex \( v \) of \( G \), the metric representation of \( v \) with respect to \( W \) is the \( k \)-vector \( r(v|W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k)) \). The set \( W \) is called a resolving set of \( G \) if \( r(u|W) = r(v|W) \) implies \( u = v \). The metric dimension of \( G \), denoted by \( \beta(G) \), is the minimum cardinality of a resolving set of \( G \).

Let \( n \geq 3 \) be an integer. A truncated wheel, denoted by \( TW_n \), is the graph with vertex set \( V(TW_n) = \{a\} \cup B \cup C \), where \( B = \{b_i : 1 \leq i \leq n\} \) and \( C = \{c_{j,k} : 1 \leq j \leq n, 1 \leq k \leq 2\} \), and edge set \( E(TW_n) = \{ab_i : 1 \leq i \leq n\} \cup \{b_ic_{i,k} : 1 \leq i \leq n, 1 \leq k \leq 2\} \cup \{c_{j,1}c_{j,2} : 1 \leq j \leq n\} \cup \{c_{j,2}c_{j+1,1} : 1 \leq j \leq n\} \), where \( c_{n+1,1} = c_{1,1} \).

In this paper, we compute the metric dimension of truncated wheels.

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Keywords: resolving set, metric dimension, truncated wheel

1 Introduction

The metric dimension problem was first introduced by Harary and Melter [5] in 1976 and independently by Slater [8] in 1988. Several authors studied this topic and published numerous results. Interested readers are also referred to [1], [2], [3], [4], [6], and [9].

Let $G = (V(G), E(G))$ be a finite, simple, and connected graph. The distance between two vertices $u$ and $v$ of $G$, denoted by $d(u, v)$, is the length of the shortest $u$-$v$ path in $G$. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the $k$-vector (ordered $k$-tuple) $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ as the metric representation of vertex $v$ with respect to $W$. The set $W$ is called a resolving set of $G$ if every two distinct vertices $u$ and $v$ satisfy $r(u|W) \neq r(v|W)$. The metric dimension of $G$, denoted by $\beta(G)$, is the minimum cardinality of a resolving set of $G$.

For a given ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of $G$, it is not difficult to see that the $i$th coordinate of $r(v|W)$ is 0 if and only if $v = w_i$. Moreover, although $W$ is treated as an ordered set, when its elements are permuted, the coordinates of $r(v|W)$ will follow correspondingly. Thus, to show that $W$ is a resolving set of $G$, it suffices to verify that $r(u|W) \neq r(v|W)$ for each pair of distinct vertices $u, v \in V(G) \setminus W$ for one particular ordering of the elements of $W$.

Let $n \geq 3$ be an integer. A truncated wheel, denoted by $TW_n$, is the graph with vertex set $V(TW_n) = \{a\} \cup B \cup C$, where $B = \{b_i : 1 \leq i \leq n\}$ and $C = \{c_{j,k} : 1 \leq j \leq n, 1 \leq k \leq 2\}$, and edge set $E(TW_n) = \{ab_i : 1 \leq i \leq n\} \cup \{b_i c_{i,k} : 1 \leq i \leq n, 1 \leq k \leq 2\} \cup \{c_{j,1} c_{j,2} : 1 \leq j \leq n\} \cup \{c_{\left \lfloor \frac{j}{2} \right \rfloor} c_{\left \lfloor \frac{j}{2} \right \rfloor + 1} : 1 \leq j \leq n\}$, where $c_{n+1,1} = c_{1,1}$. This new graph species was introduced by Lee [7].

In the succeeding computations, additions on the subscript of vertex $b_i$ and the first subscript of vertex $c_{j,k}$ are taken modulo $n$, while that of the second subscript of $c_{j,k}$ is taken modulo 2.

In this paper, by proving several propositions and lemmas, we completely compute the metric dimension of all truncated wheels. We end the paper with a conjecture on the metric dimension of a variant of truncated wheels.

2 Main Results

Chartrand, et. al. [4] showed that, for a connected graph $G$ of order $n$, $\beta(G) = 1$ if and only if $G$ is a path of order $n$. It follows that $\beta(TW_n) \geq 2$ for every integer $n \geq 3$. 
It is not difficult to check that $S = \{b_1, b_2\}$ and $T = \{c_{1,1}, c_{2,2}\}$ are resolving sets of $TW_3$ and $TW_5$, respectively. Thus, we have the following proposition.

**Proposition 2.1.** $\beta(TW_3) = \beta(TW_5) = 2$

**Proposition 2.2.** $\beta(TW_4) = 3$

*Proof.* It is not difficult to check that the set $S = \{b_1, b_2, b_3\}$ is a resolving set of $TW_4$. We show that there exists no resolving set $S$ of $TW_4$ with order 2.

Consider an ordered subset $S$ of $V(TW_4)$ with order 2. It suffices to consider five possibilities for $S$.

**Case 1.** If $S = \{a, b\}$ for some $b \in B$, then $r(b_k|S) = (1, 2)$ for each $b_k \neq b$ in $B$.

**Case 2.** If $S = \{b_1, b_j\}$, then $r(b_k|S) = (2, 2)$ for every $b_k \in B \setminus S$.

**Case 3.** If $S = \{a, c\}$ for some $c = c_{i,j}$, then $r(b_{i+j}|S) = r(b_{i,j+1}|S) = (1, 3)$.

**Case 4.** Suppose both vertices in $S$ are from $C$.

**Subcase 4.1.** If $S = \{c_{i,1}, c_{i+2,2}\}$ or $S = \{c_{i,1}, c_{i+1,2}\}$, then $r(b_{i+3}|S) = r(c_{i+3,1}|S) = (2, 3)$.

**Subcase 4.2.** If $S = \{c_{i,1}, c_{i+2,2}\}$, then $r(a|S) = r(b_{i+3}|S) = (2, 2)$.

**Subcase 4.3.** If $S = \{c_{i,1}, c_{i+3,2}\}$, then $r(b_{i+1}|S) = r(b_{i+2}|S) = (3, 3)$.

**Subcase 4.4.** If $S = \{c_{i,j}, c_{i+1,j}\}$, then $r(b_{i+j+1}|S) = r(c_{i+1,j+1}|S) = (3, 3)$.

**Subcase 4.5.** If $S = \{c_{i,j}, c_{i+2,j}\}$, then $r(a|S) = r(c_{i+1,j}|S) = (2, 2)$.

**Subcase 4.6.** If $S = \{c_{i,j}, c_{i+3,j}\}$, then $r(b_{i+j}|S) = r(c_{i+j,j+1}|S) = (3, 3)$.

**Case 5.** Suppose $S = \{b, c\}$ for some $b = b_i$ and $c \in C$.

**Subcase 5.1.** If $c = c_{i,1}$ or $c_{i+3,2}$, then $r(b_{i+1}|S) = r(b_{i+2}|S) = (2, 3)$.

**Subcase 5.2.** If $c = c_{i,2}$ or $c_{i+1,1}$, then $r(b_{i+2}|S) = r(b_{i+3}|S) = (2, 3)$.

**Subcase 5.3.** If $c = c_{i+1,2}$, then $r(b_{i+1}|S) = r(c_{i+1,1}|S) = (2, 1)$.

**Subcase 5.4.** If $c = c_{i+2,1}$, then $r(b_{i+1}|S) = r(c_{i+1,1}|S) = (2, 2)$.

**Subcase 5.5.** If $c = c_{i+2,2}$, then $r(b_{i+3}|S) = r(c_{i+3,2}|S) = (2, 2)$.

**Subcase 5.6.** If $c = c_{i+3,1}$, then $r(b_{i+3}|S) = r(c_{i+3,2}|S) = (2, 1)$. 
The existence of two vertices with the same metric representation with respect to $S$ in each of the cases and subcases implies that there cannot exist a resolving set of $TW_4$ with order 2. Therefore, we conclude that $\beta(TW_4) = 3$. \hfill \qed

Lemma 2.3. Let $n \geq 6$ be an integer. Then $\beta(TW_n) \leq \lceil n/2 \rceil$.

Proof. Let $\alpha = \lceil n/2 \rceil$. We claim that $S = \{c_{1,1}, c_{3,1}, c_{5,1}, \ldots, c_{2\alpha-1,1}\}$ is a resolving set of $TW_n$. Note that $|S| = \alpha$.

We separate the cases when $n$ is even and when $n$ is odd. We enumerate the distinct metric representations of $v$ with respect to $S$ for all $v \in V(TW_n) \setminus S$.

**Case 1.** Suppose $n$ is even. Then the metric representations of the vertices of $TW_n$ are as follows.

(1.1) $r(a|S) = (2, 2, \ldots, 2)$

(1.2) For each $i = 1, 3, 5, \ldots, n-1$, $r(b_i|S)$ has $(\alpha - 1)$ 3’s and one 1.

(1.3) For each $i = 2, 4, 6, \ldots, n$, $r(b_i|S)$ has $(\alpha - 1)$ 3’s and one 2.

(1.4) For each $i = 2, 4, 6, \ldots, n$, $r(c_{i,1}|S)$ has $(\alpha - 2)$ 4’s and two 2’s.

(1.5) For each $i = 1, 3, 5, \ldots, n-1$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4’s, one 1, and one 3, where, in particular, 1 is located in the $((i+1)/2)$th coordinate.

(1.6) For each $i = 2, 4, 6, \ldots, n$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4’s, one 1, and one 3, where, in particular, 1 is located in the $((i/2) + 1)$th coordinate.

**Case 2.** Suppose $n$ is odd. Then we have the following metric representations of the vertices of $TW_n$.

(2.1) $r(a|S) = (2, 2, \ldots, 2)$

(2.2) For each $i = 1, 3, 5, \ldots, n - 2$, $r(b_i|S)$ has $(\alpha - 1)$ 3’s and one 1. Moreover, we have $r(b_n|S) = (2, 3, 3, \ldots, 3)$.

(2.3) For each $i = 2, 4, 6, \ldots, n - 3$, $r(b_i|S)$ has $(\alpha - 1)$ 3’s and one 2. Moreover, we have $r(b_{n-1}|S) = (3, 3, \ldots, 3)$.

(2.4) $r(c_{n,1}|S) = (2, 4, 4, \ldots, 4)$

(2.5) For each $i = 2, 4, 6, \ldots, n - 3$, $r(c_{i,1}|S)$ has $(\alpha - 2)$ 4’s and two 2’s. Moreover, we have $r(c_{n-1,1}|S) = (4, 4, \ldots, 4, 2)$. 
(2.6) For each \( i = 1, 3, 5, \ldots, n - 4 \), \( r(c_i, 2|S) \) has \((\alpha - 2)\) 4's, one 1, and one 3, where, in particular, 1 and 3 are located in the \((i + 1)/2)\)th and \(((i + 3)/2)\)th coordinates, respectively. Moreover, we have \( r(c_{n-2}, 2|S) = (4, 4, \ldots, 4, 1) \) and \( r(c_{n, 2}|S) = (1, 4, 4, \ldots, 4) \).

(2.7) For each \( i = 2, 4, 6, \ldots, n - 3 \), \( r(c_i, 2|S) \) has \((\alpha - 2)\) 4's, one 1, and one 3, where, in particular, 1 is located in the \((i/2) + 1)\)th coordinate. Moreover, we have \( r(c_{n-1, 2}|S) = (3, 4, 4, \ldots, 4, 3) \).

It is not difficult to verify that all these metric representations with respect to \( S \) are distinct.

The following lemma is not difficult to prove, and so we omit its proof.

**Lemma 2.4.** Let \( n \geq 3 \). If \( S \) satisfies one of the following conditions:

(i) \( S \subset B \) with \( 1 \leq |S| \leq n - 2 \), or

(ii) \( S = \{a\} \cup S_B \), where \( S_B \subset B \) with \( 1 \leq |S_B| \leq n - 2 \),

then all vertices in \( B \setminus S \) have the same metric representation with respect to \( S \).

**Lemma 2.5.** Let \( n \geq 6 \). If \( S \) satisfies one of the following conditions:

(i) \( S \subset C \) with \( 1 \leq |S| \leq \lfloor n/2 \rfloor - 1 \), or

(ii) \( S = \{a\} \cup S_C \), where \( S_C \subset C \) with \( 1 \leq |S_C| \leq \lfloor n/2 \rfloor - 2 \),

then there exist two vertices in \( B \) that are not adjacent to any vertex in \( S \cap C \) and that have the same metric representation with respect to \( S \).

**Proof.** We prove the result when \( S \) satisfies condition (i). The result then follows immediately when \( S \) satisfies condition (ii).

Let \( \alpha = |S| \), and \( B' \) be the set of vertices in \( B \) that are not adjacent to any vertex in \( S \). Then \( |B'| \geq n - \alpha \geq 4 \). We show that two vertices in \( B' \) have metric representations with respect to \( S \) that consist of \( \alpha \) 3’s.

Suppose that only at most one vertex in \( B' \) has metric representation with respect to \( S \) consisting of \( \alpha \) 3’s. Then at least \(|B'| - 1\) vertices in \( B' \) have distance 2 from a vertex in \( S \).

Since every vertex in \( B \) has exactly two vertices in \( C \) of distance 2, it follows that

\[
|S| \geq |B'| - 1 \geq n - \alpha - 1 \geq n - \left( \frac{n}{2} - 1 \right) - 1 = \frac{n}{2},
\]

which is a contradiction.

**Lemma 2.6.** Let \( n \geq 6 \). If \( S \) satisfies one of the following conditions:
(i) $S = S_B \cup S_C$, where $S_B \subset B$, $S_C \subset C$, $|S_B| \geq 1$, $|S_C| \geq 1$, and $|S| \leq \lfloor n/2 \rfloor - 1$, or

(ii) $S = \{a\} \cup S_B \cup S_C$, where $S_B$ and $S_C$ are similarly defined as in (i), except that $|S_B| + |S_C| \leq \lfloor n/2 \rfloor - 2$,

then there exist two vertices in $B \setminus S_B$ with the same metric representation with respect to $S$.

Proof. We prove the result when condition (i) holds. Then, because $d(b, a) = 1$ for all $b \in B$, the result immediately follows when condition (ii) holds.

Let $S' = S_B' \cup S_C$, where $S_B'$ is the set obtained from $S_B$ by replacing each $b_i \in S_B$ by $c_{i,1}$. Clearly, $S_B' \subset C$ and $|S'| \leq |S| \leq \lfloor n/2 \rfloor - 1$. By Lemma 2.5, there exist two vertices $u$ and $v$ in $B$ that are not adjacent to any vertex in $S'$ and that have a common metric representation with respect to $S'$. Because $d(b_i, b_j) = 2$ for every pair of distinct vertices $b_i$ and $b_j$ in $B$, these vertices $u$ and $v$ also have the same metric representation with respect to $S$. \hfill \Box

The following lemma summarizes Lemmas 2.4, 2.5, and 2.6.

**Lemma 2.7.** Let $n \geq 6$ be an integer. Then there exists no resolving set of $TW_n$ with order less than $\lfloor n/2 \rfloor$; that is, $\beta(TW_n) \geq \lfloor n/2 \rfloor$.

By Propositions 2.1 and 2.2 and Lemmas 2.3 and 2.7, we have the main theorem of the paper.

**Theorem 2.8.** Let $n \geq 3$ be an integer. Then

$$\beta(TW_n) = \begin{cases} 
2 & \text{if } n = 3 \\
3 & \text{if } n = 4 \\
\lfloor n/2 \rfloor & \text{if } n \geq 5.
\end{cases}$$

3 A Variant of $TW_n$

Let $n \geq 3$ be an integer. Define $TW_n^*$ as the graph with vertex set $V(TW_n^*) = V(TW_n)$ and edge set $E(TW_n^*) = E(TW_n) \cup \{b_ib_{i+1} : 1 \leq i \leq n\}$, where $b_{n+1} = b_1$.

It is easy to check that $S = \{b_1, b_2, c_{3,1}\}$ is a resolving set of $TW_n^*$. It is also not difficult to show that $\beta(TW_3^*) \geq 3$. It follows that $\beta(TW_3^*) = 3$.

Moreover, it can be verified that $\beta(TW_4^*) = \beta(TW_5^*) = 2$ and $\beta(TW_6^*) = 3$.

With the initial values above, we end the paper with a conjecture on the metric dimension of $TW_n^*$. 
Conjecture 3.1. Let $n \geq 3$ be an integer. Then

$$\beta(TW^*_n) = \begin{cases} 
3 & \text{if } n = 3 \text{ or } n = 6 \\
2 & \text{if } n = 4 \text{ or } n = 5 \\
\lceil n/2 \rceil - 1 & \text{if } n \geq 7.
\end{cases}$$

References


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