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Computing the Metric Dimension of Truncated Wheels

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Abstract

For an ordered subset $W = \{w_1, w_2, w_3, \ldots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$. The set W is called a resolving set of G if $r(u|W) = r(v|W)$ implies $u = v$. The metric dimension of G, denoted by $\beta(G)$, is the minimum cardinality of a resolving set of G.

Let $n \geq 3$ be an integer. A truncated wheel, denoted by TW_n, is the graph with vertex set $V(TW_n) = \{a\} \cup B \cup C$, where $B = \{b_i : 1 \leq i \leq n\}$ and $C = \{c_{j,k} : 1 \leq j \leq n, 1 \leq k \leq 2\}$, and edge set $E(TW_n) = \{ab_i : 1 \leq i \leq n\} \cup \{b_i c_{i,k} : 1 \leq i \leq n, 1 \leq k \leq 2\} \cup \{c_{j,1}c_{j,2} : 1 \leq j \leq n\} \cup \{c_{j,2}c_{j+1,1} : 1 \leq j \leq n\}$, where $c_{n+1,1} = c_{1,1}$.

In this paper, we compute the metric dimension of truncated wheels.

Mathematics Subject Classification: 05C12

Keywords: resolving set, metric dimension, truncated wheel

1 Introduction

The metric dimension problem was first introduced by Harary and Melter [5] in 1976 and independently by Slater [8] in 1988. Several authors studied this topic and published numerous results. Interested readers are also referred to [1], [2], [3], [4], [6], and [9].

Let $G = (V(G), E(G))$ be a finite, simple, and connected graph. The *distance* between two vertices u and v of G, denoted by $d(u, v)$, is the length of the shortest u-v path in G. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector (ordered k-tuple) $r(v|W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$ as the *metric representation of vertex v with respect to W*. The set W is called a *resolving set of G* if every two distinct vertices u and v satisfy $r(u|W) \neq r(v|W)$. The *metric dimension of G*, denoted by $\beta(G)$, is the minimum cardinality of a resolving set of G.

For a given ordered set $W = \{w_1, w_2, , w_3 \ldots, w_k\}$ of vertices of G, it is not difficult to see that the ith coordinate of $r(v|W)$ is 0 if and only if $v = w_i$. Moreover, although W is treated as an ordered set, when its elements are permuted, the coordinates of $r(v|W)$ will follow correspondingly. Thus, to show that W is a resolving set of G, it suffices to verify that $r(u|W) \neq r(v|W)$ for each pair of distinct vertices $u, v \in V(G)\backslash W$ for one particular ordering of the elements of W.

Let $n \geq 3$ be an integer. A *truncated wheel*, denoted by TW_n, is the graph with vertex set $V(TW_n) = \{a\} \cup B \cup C$, where $B = \{b_i : 1 \leq i \leq n\}$ and $C = \{c_{j,k} : 1 \leq j \leq n, 1 \leq k \leq 2\}$, and edge set $E(TW_n) = \{ab_i : 1 \leq i \leq n\} \cup \{b_i c_{i,k} : 1 \leq i \leq n, 1 \leq k \leq 2\} \cup \{c_{j,1}c_{j,2} : 1 \leq j \leq n\} \cup \{c_{j,2}c_{j+1,1} : 1 \leq j \leq n\}$, where $c_{n+1,1} = c_{1,1}$. This new graph specie was introduced by Lee [7].

In the succeeding computations, additions on the subscript of vertex b_i and the first subscript of vertex $c_{j,k}$ are taken modulo n, while that of the second subscript of $c_{j,k}$ is taken modulo 2.

In this paper, by proving several propositions and lemmas, we completely compute the metric dimension of all truncated wheels. We end the paper with a conjecture on the metric dimension of a variant of truncated wheels.

2 Main Results

Chartrand, et. al. [4] showed that, for a connected graph G of order n, $\beta(G) = 1$ if and only if G is a path of order n. It follows that $\beta(TW_n) \geq 2$ for every integer $n \geq 3$.

It is not difficult to check that $S = \{b_1, b_2\}$ and $T = \{c_{1,1}, c_{2,2}\}$ are resolving sets of TW_3 and TW_5, respectively. Thus, we have the following proposition.

Proposition 2.1. $\beta(TW_3) = \beta(TW_5) = 2$

Proposition 2.2. $\beta(TW_4) = 3$

Proof. It is not difficult to check that the set $S = \{b_1, b_2, b_3\}$ is a resolving set of TW_4. We show that there exists no resolving set S of TW_4 with order 2.

Consider an ordered subset S of $V(TW_4)$ with order 2. It suffices to consider five possibilities for S.

CASE 1. If $S = \{a, b\}$ for some $b \in B$, then $r(b_k|S) = (1, 2)$ for each $b_k \neq b$ in B.

CASE 2. If $S = \{b_i, b_j\}$, then $r(b_k|S) = (2, 2)$ for every $b_k \in B \backslash S$.

CASE 3. If $S = \{a, c\}$ for some $c = c_{i,j}$, then $r(b_{i+j}|S) = r(b_{i+j+1}|S) = (1, 3)$.

CASE 4. Suppose both vertices in S are from C.

SUBCASE 4.1. If $S = \{c_{i,1}, c_{i,2}\}$ or $S = \{c_{i,1}, c_{i+1,2}\}$, then $r(b_{i+3}|S) = r(c_{i+3,1}|S) = (2, 3)$.

SUBCASE 4.2. If $S = \{c_{i,1}, c_{i+2,2}\}$, then $r(a|S) = r(b_{i+3}|S) = (2, 2)$.

SUBCASE 4.3. If $S = \{c_{i,1}, c_{i+3,2}\}$, then $r(b_{i+1}|S) = r(b_{i+2}|S) = (3, 3)$.

SUBCASE 4.4. If $S = \{c_{i,j}, c_{i+1,j}\}$, then $r(b_{i+j+1}|S) = r(c_{i+j+1,j+1}|S) = (3, 3)$.

SUBCASE 4.5. If $S = \{c_{i,j}, c_{i+2,j}\}$, then $r(a|S) = r(c_{i+1,j}|S) = (2, 2)$.

SUBCASE 4.6. If $S = \{c_{i,j}, c_{i+3,j}\}$, then $r(b_{i+j}|S) = r(c_{i+j,j+1}|S) = (3, 3)$.

CASE 5. Suppose $S = \{b, c\}$ for some $b = b_i$ and $c \in C$.

SUBCASE 5.1. If $c = c_{i,1}$ or $c_{i+3,2}$, then $r(b_{i+1}|S) = r(b_{i+2}|S) = (2, 3)$.

SUBCASE 5.2. If $c = c_{i,2}$ or $c_{i+1,1}$, then $r(b_{i+2}|S) = r(b_{i+3}|S) = (2, 3)$.

SUBCASE 5.3. If $c = c_{i+1,2}$, then $r(b_{i+1}|S) = r(c_{i+1,1}|S) = (2, 1)$.

SUBCASE 5.4. If $c = c_{i+2,1}$, then $r(b_{i+1}|S) = r(c_{i+1,1}|S) = (2, 2)$.

SUBCASE 5.5. If $c = c_{i+2,2}$, then $r(b_{i+3}|S) = r(c_{i+3,2}|S) = (2, 2)$.

SUBCASE 5.6. If $c = c_{i+3,1}$, then $r(b_{i+3}|S) = r(c_{i+3,2}|S) = (2, 1)$.

The existence of two vertices with the same metric representation with respect to S in each of the cases and subcases implies that there cannot exist a resolving set of TW_4 with order 2. Therefore, we conclude that $\beta(TW_4) = 3$. □

Lemma 2.3. *Let $n \geq 6$ be an integer. Then $\beta(TW_n) \leq \lfloor n/2 \rfloor$.*

Proof. Let $\alpha = \lfloor n/2 \rfloor$. We claim that $S = \{c_{1,1}, c_{3,1}, c_{5,1}, \ldots, c_{2\alpha-1,1}\}$ is a resolving set of TW_n. Note that $|S| = \alpha$.

We separate the cases when n is even and when n is odd. We enumerate the distinct metric representations of v with respect to S for all $v \in V(TW_n)\backslash S$.

Case 1. Suppose n is even. Then the metric representations of the vertices of TW_n are as follows.

(1.1) $r(a|S) = (2, 2, \ldots, 2)$

(1.2) For each $i = 1, 3, 5, \ldots, n-1$, $r(b_i|S)$ has $(\alpha - 1)$ 3's and one 1.

(1.3) For each $i = 2, 4, 6, \ldots, n$, $r(b_i|S)$ has $(\alpha - 1)$ 3's and one 2.

(1.4) For each $i = 2, 4, 6, \ldots, n$, $r(c_{i,1}|S)$ has $(\alpha - 2)$ 4's and two 2's.

(1.5) For each $i = 1, 3, 5, \ldots, n-1$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4's, one 1, and one 3, where, in particular, 1 is located in the $((i+1)/2)$th coordinate.

(1.6) For each $i = 2, 4, 6, \ldots, n$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4's, one 1, and one 3, where, in particular, 1 is located in the $((i/2) + 1)$th coordinate.

Case 2. Suppose n is odd. Then we have the following metric representations of the vertices of TW_n.

(2.1) $r(a|S) = (2, 2, \ldots, 2)$

(2.2) For each $i = 1, 3, 5, \ldots, n-2$, $r(b_i|S)$ has $(\alpha - 1)$ 3's and one 1. Moreover, we have $r(b_n|S) = (2, 3, 3, \ldots, 3)$.

(2.3) For each $i = 2, 4, 6, \ldots, n-3$, $r(b_i|S)$ has $(\alpha - 1)$ 3's and one 2. Moreover, we have $r(b_{n-1}|S) = (3, 3, \ldots, 3)$.

(2.4) $r(c_{n,1}|S) = (2, 4, 4, \ldots, 4)$

(2.5) For each $i = 2, 4, 6, \ldots, n-3$, $r(c_{i,1}|S)$ has $(\alpha - 2)$ 4's and two 2's. Moreover, we have $r(c_{n-1,1}|S) = (4, 4, \ldots, 4, 2)$.

(2.6) For each $i = 1, 3, 5, \ldots, n-4$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4's, one 1, and one 3, where, in particular, 1 and 3 are located in the $((i+1)/2)$th and $((i+3)/2)$th coordinates, respectively. Moreover, we have $r(c_{n-2,2}|S) = (4, 4, \ldots, 4, 1)$ and $r(c_{n,2}|S) = (1, 4, 4, \ldots, 4)$.

(2.7) For each $i = 2, 4, 6, \ldots, n-3$, $r(c_{i,2}|S)$ has $(\alpha - 2)$ 4's, one 1, and one 3, where, in particular, 1 is located in the $((i/2)+1)$th coordinate. Moreover, we have $r(c_{n-1,2}|S) = (3, 4, 4, \ldots, 4, 3)$.

It is not difficult to verify that all these metric representations with respect to S are distinct. □

The following lemma is not difficult to prove, and so we omit its proof.

Lemma 2.4. *Let $n \geq 3$. If S satisfies one of the following conditions:*

(i) $S \subset B$ with $1 \leq |S| \leq n-2$, or

(ii) $S = \{a\} \cup S_B$, where $S_B \subset B$ with $1 \leq |S_B| \leq n-2$,

then all vertices in $B \backslash S$ have the same metric representation with respect to S.

Lemma 2.5. *Let $n \geq 6$. If S satisfies one of the following conditions:*

(i) $S \subset C$ with $1 \leq |S| \leq \lfloor n/2 \rfloor - 1$, or

(ii) $S = \{a\} \cup S_C$, where $S_C \subset C$ with $1 \leq |S_C| \leq \lfloor n/2 \rfloor - 2$,

then there exist two vertices in B that are not adjacent to any vertex in $S \cap C$ and that have the same metric representation with respect to S.

Proof. We prove the result when S satisfies condition (i). The result then follows immediately when S satisfies condition (ii).

Let $\alpha = |S|$, and B' be the set of vertices in B that are not adjacent to any vertex in S. Then $|B'| \geq n - \alpha \geq 4$. We show that two vertices in B' have metric representations with respect to S that consist of α 3's.

Suppose that only at most one vertex in B' has metric representation with respect to S consisting of α 3's. Then at least $|B'| - 1$ vertices in B' have distance 2 from a vertex in S.

Since every vertex in B has exactly two vertices in C of distance 2, it follows that
$$|S| \geq |B'| - 1 \geq n - \alpha - 1 \geq n - \left(\frac{n}{2} - 1\right) - 1 = \frac{n}{2},$$
which is a contradiction. □

Lemma 2.6. *Let $n \geq 6$. If S satisfies one of the following conditions:*

(i) $S = S_B \cup S_C$, *where* $S_B \subset B$, $S_C \subset C$, $|S_B| \geq 1$, $|S_C| \geq 1$, *and* $|S| \leq \lfloor n/2 \rfloor - 1$, *or*

(ii) $S = \{a\} \cup S_B \cup S_C$, *where* S_B *and* S_C *are similarly defined as in (i), except that* $|S_B| + |S_C| \leq \lfloor n/2 \rfloor - 2$,

then there exist two vertices in $B \backslash S_B$ *with the same metric representation with respect to* S.

Proof. We prove the result when condition (i) holds. Then, because $d(b, a) = 1$ for all $b \in B$, the result immediately follows when condition (ii) holds.

Let $S' = S'_B \cup S_C$, where S'_B is the set obtained from S_B by replacing each $b_i \in S_B$ by $c_{i,1}$. Clearly, $S'_B \subset C$ and $|S'| \leq |S| \leq \lfloor n/2 \rfloor - 1$. By Lemma 2.5, there exist two vertices u and v in B that are not adjacent to any vertex in S' and that have a common metric representation with respect to S'. Because $d(b_i, b_j) = 2$ for every pair of distinct vertices b_i and b_j in B, these vertices u and v also have the same metric representation with respect to S. □

The following lemma summarizes Lemmas 2.4, 2.5, and 2.6.

Lemma 2.7. *Let* $n \geq 6$ *be an integer. Then there exists no resolving set of* TW_n *with order less than* $\lfloor n/2 \rfloor$; *that is,* $\beta(TW_n) \geq \lfloor n/2 \rfloor$.

By Propositions 2.1 and 2.2 and Lemmas 2.3 and 2.7, we have the main theorem of the paper.

Theorem 2.8. *Let* $n \geq 3$ *be an integer. Then*

$$\beta(TW_n) = \begin{cases} 2 & \text{if } n = 3 \\ 3 & \text{if } n = 4 \\ \lfloor n/2 \rfloor & \text{if } n \geq 5. \end{cases}$$

3 A Variant of TW_n

Let $n \geq 3$ be an integer. Define TW_n^* as the graph with vertex set $V(TW_n^*) = V(TW_n)$ and edge set $E(TW_n^*) = E(TW_n) \cup \{b_i b_{i+1} : 1 \leq i \leq n\}$, where $b_{n+1} = b_1$.

It is easy to check that $S = \{b_1, b_2, c_{3,1}\}$ is a resolving set of TW_3^*. It is also not difficult to show that $\beta(TW_3^*) \geq 3$. It follows that $\beta(TW_3^*) = 3$.

Moreover, it can be verified that $\beta(TW_4^*) = \beta(TW_5^*) = 2$ and $\beta(TW_6^*) = 3$.

With the initial values above, we end the paper with a conjecture on the metric dimension of TW_n^*.

Conjecture 3.1. *Let $n \geq 3$ be an integer. Then*

$$\beta(TW_n^*) = \begin{cases} 3 & \text{if } n = 3 \text{ or } n = 6 \\ 2 & \text{if } n = 4 \text{ or } n = 5 \\ \lceil n/2 \rceil - 1 & \text{if } n \geq 7. \end{cases}$$

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