2015

Cauchy Extension of $M\alpha$-Integral and Absolute $M\alpha$-Integrability

Ian June L. Garces
Abraham P. Racca

Follow this and additional works at: https://archium.ateneo.edu/mathematics-faculty-pubs

Part of the Mathematics Commons
Cauchy Extension of $M_\alpha$-Integral and Absolute $M_\alpha$-Integrability

I. J. L. Garces

Department of Mathematics
Ateneo de Manila University
1108 Loyola Heights, Quezon City
The Philippines

Abraham P. Racca

Mathematics and Computer Science Department
Adventist University of the Philippines
4118 Silang, Cavite, The Philippines

Copyright © 2015 I. J. L. Garces and Abraham P. Racca. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Park, Ryu, and Lee recently introduced a Henstock-type integral, which lies between the Mcshane and the Henstock integrals. This paper proves the closure property of this new integral under Cauchy extension, and presents a characterization on absolute $M_\alpha$-integrability.

Mathematics Subject Classification: 26A39

Keywords: $M_\alpha$-integral, Cauchy extension, absolute $M_\alpha$-integrable

1 Introduction

Park, Ryu, and Lee [4] recently defined a Henstock-type integral, which they call $M_\alpha$-integral. Several properties and convergence theorems of the integral were established in [1] and [4]. Most of them paralleled the usual properties
of Henstock integral, including the Saks-Henstock Lemma [4, Lemma 2.5]. Moreover, by providing examples, it was also shown in [4] that $M_\alpha$-integral lies strictly between the McShane integral (which is known to be equivalent to the Lebesgue integral (see [2])) and the Henstock integral.

Let $\alpha > 0$ be a constant, and $I = [a, b]$ a non-degenerate closed and bounded interval in $\mathbb{R}$.

1. A *partial division* $D$ of $I$ is a finite collection of interval-point pairs $([u, v], \xi)$ such that the closed intervals $[u, v]$ are non-overlapping, $\bigcup [u, v] \subseteq I$, and $\xi \in I$. If $\bigcup [u, v] = I$, we call the division $D$ simply a *division*.

2. A positive function defined on $I$ is called a *gauge* on $I$.

3. Let $\delta$ be a gauge on $I$, and $D = \{([u, v], \xi)\}$ a partial division of $I$. If $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$ for all $([u, v], \xi) \in D$, then we say that $D$ is a $\delta$-fine McShane partial division. Moreover, if $D$ is a McShane partial division such that $\xi \in [u, v]$ for all $([u, v], \xi) \in D$, then $D$ is called a $\delta$-fine Henstock partial division.

4. A McShane division $D = \{([u, v], \xi)\}$ of $I$ is said to be an $M_\alpha$-division if

$$\sum_{([u, v], \xi) \in D} \text{dist}(\xi, [u, v]) < \alpha,$$

where $\text{dist}(x, J) = \inf\{|y - x| : y \in J\}$.

5. Let $D = \{([u, v], \xi)\}$ be a partial division on $I$, and $f$ a real-valued function defined on $I$. We write

$$S(f, D) = \sum_{([u, v], \xi) \in D} f(\xi)(v - u).$$

With these terms and notations, the definition of $M_\alpha$-integrability can now be presented.

**Definition 1.1** ([4, Definition 2.1]). A function $f : [a, b] \to \mathbb{R}$ is $M_\alpha$-integrable if there exists a real number $A$ such that, for each $\epsilon > 0$, there is a gauge $\delta$ on $[a, b]$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta$-fine $M_\alpha$-division $D$ of $[a, b]$. Here, $A$ is called the $M_\alpha$-integral of $f$ on $[a, b]$, and we write $A = \int_a^b f$.

Moreover, $f$ is $M_\alpha$-integrable on $E \subseteq [a, b]$ if $f \chi_E$ is $M_\alpha$-integrable on $[a, b]$, where $\chi_E$ is the characteristic function over $E$, and we write $\int_a^b f \chi_E = \int_E f$.

This paper proves the closure property of this new integral under Cauchy extension (Theorem 2.1) and a characterization on absolute $M_\alpha$-integrability (Theorem 3.3).


2 Cauchy Extension

The following theorem shows that $M_\alpha$-integral is closed under Cauchy extension, a property that is not valid for McShane integral.

**Theorem 2.1 (Cauchy Extension).** Let $f : [a, b] \to \mathbb{R}$ be $M_\alpha$-integrable on $[c, b]$ for each $c \in (a, b)$. Then $f$ is $M_\alpha$-integrable on $[a, b]$ if and only if $\lim_{c \to a^+} \int_c^b f$ exists. In this case, we have

$$
\int_a^b f = \lim_{c \to a^+} \int_c^b f.
$$

**Proof.** Suppose that $f$ is $M_\alpha$-integrable on $[a, b]$, and let $\varepsilon > 0$ be given. Then there exists a gauge $\delta$ on $[a, b]$ such that if $D$ is a $\delta$-fine $M_\alpha$-division of $[a, b]$, then

$$
|S(f, D) - \int_a^b f| < \varepsilon.
$$

Moreover, for each $c \in (a, b)$, there exists a gauge $\delta_c$ on $[c, b]$ such that if $D_c$ is a $\delta_c$-fine $M_\alpha$-division of $[c, b]$, then

$$
|S(f, D_c) - \int_c^b f| < \varepsilon.
$$

For each $c \in (a, b)$, we may assume that $\delta_c(x) \leq \delta(x)$ for all $x \in [c, b]$.

Choose $s \in (a, a + \delta(a))$ such that $|f(a)|(s - a) < \varepsilon$. Let $D_s$ be a $\delta_s$-fine $M_\alpha$-division of $[s, b]$. Then $D = D_s \cup \{[a, s], a\}$ is a $\delta$-fine $M_\alpha$-division of $[a, b]$. Hence, we have

$$
\left| \int_a^b f - \int_s^b f \right| \leq \left| \int_a^b f - S(f, D) \right| + \left| S(f, D_s) - \int_s^b f \right| + |f(a)|(s - a) < 3\varepsilon.
$$

Since $s$ is chosen arbitrarily close to $a$, we obtain $\lim_{c \to a^+} \int_c^b f$ exists and is equal to $\int_a^b f$.

Conversely, suppose that $\lim_{c \to a^+} \int_c^b f = A$ exists, and let $\{x_0, x_1, x_2, \ldots\}$ be a strictly decreasing sequence of real numbers such that $x_0 = b$ and $x_n \to a$. Then $f$ is $M_\alpha$-integrable on $[x_n, x_{n-1}]$ for every $n \in \mathbb{Z}^+$.

Let $\varepsilon > 0$ be given. For each integer $n \geq 1$, choose a gauge $\delta_n$ on $[x_n, x_{n-1}]$ such that if $D_n$ is a $\delta_n$-fine $M_\alpha$-division of $[x_n, x_{n-1}]$, then

$$
|S(f, D_n) - \int_{x_n}^{x_{n-1}} f| < \frac{\varepsilon}{2^n}.
$$

Furthermore, choose an integer $N > 0$ such that

$$
\left| \int_t^b f - A \right| < \varepsilon \quad \text{and} \quad |f(a)|(t - a) < \varepsilon
$$
for every $t \in (a, x_N]$.

Define a gauge $\delta$ on $[a, b]$ as follows:

$$
\delta(x) = \begin{cases} 
x_N - x & \text{if } x = a \\
\min\{\delta_1(x), x - x_1\} & \text{if } x \in (x_1, x_0] \\
\min\{\delta_n(x), x - x_n, x_{n-1} - x\} & \text{if } x \in (x_n, x_{n-1}), n \geq 2 \\
\min\{\delta_n(x), \delta_{n+1}(x), x_{n-1} - x, x - x_{n+1}\} & \text{if } x = x_n, n \geq 1.
\end{cases}
$$

Let $D$ be a $\delta$-fine $M_\alpha$-division of $[a, b]$. Then

$$
D = \{([a, t], a)\} \cup D_t \cup \bigcup_{i=1}^m D_i,
$$

where $t \in (x_{m+1}, x_m]$ for some integer $m \geq N$, $D_t$ a $\delta_{m+1}$-fine $M_\alpha$-division of $[t, x_m] \subset (x_{m+1}, x_m]$, and $D_i$ a $\delta_i$-fine $M_\alpha$-division of $[x_i, x_{i-1}]$. Thus, by Saks-Henstock Lemma, we have

$$
|S(f, D) - A| \leq |f(a)|(t - a) + \left| S(f, D_t) - \int_{x_1}^{x_m} f \right| + \sum_{i=1}^m \left| S(f, D_i) - \int_{x_{i-1}}^{x_i} f \right| + \left| \int_{x_{m+1}}^{x_m} f - A \right| < \epsilon + \frac{\epsilon}{2^{m+1}} + \sum_{i=1}^\infty \frac{\epsilon}{2^i} + \epsilon < 4\epsilon.
$$

This ends the proof of the theorem. □

**Corollary 2.2.** Let $f : [a, b] \to \mathbb{R}$ be $M_\alpha$-integrable on every $[c, d] \subset (a, b)$. Then $f$ is $M_\alpha$-integrable on $[a, b]$ if and only if $\lim_{d \to b^-} \int_c^d f$ exists. In this case, we have

$$
\int_a^b f = \lim_{d \to b^-} \int_c^d f.
$$

### 3 Absolute $M_\alpha$-Integrability

Recall that an integral theory is *absolute* if a function $f$ is integrable (in the sense of that theory) if and only if $|f|$ is also integrable. Both Riemann and McShane integrals are absolute (see [2] or [5]), but Henstock integral is known to be not (see [3]).
Example 3.1. Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series. Define a function \( f : [0,1] \to \mathbb{R} \) as follows:

\[
f(x) = \begin{cases} 
2^n a_n & \text{if } x \in \left( \frac{1}{2^n}, \frac{1}{2^n-1} \right], \ n \in \mathbb{Z}^+ \\
0 & \text{if } x = 0.
\end{cases}
\]

Clearly, \( f \) is \( M_\alpha \)-integrable on \([c,1]\) for each \( c \in (0,1) \). Thus, by Theorem 2.1, we have

\[
\int_{0}^{1} f = \lim_{c \to 0^+} \int_{c}^{1} f = \sum_{n=1}^{\infty} a_n.
\]

However, it is not difficult to see that \( |f| \) would be \( M_\alpha \)-integrable if and only if the given series is absolutely convergent.

In general, due to the closure under Cauchy extension, it is possible for a function \( f \) to be \( M_\alpha \)-integrable on \([a,b]\), but \( |f| \) is not. We have the following proposition.

**Proposition 3.2.** The \( M_\alpha \)-integral is not absolute.

Let a function \( f : [a,b] \to \mathbb{R} \) be \( M_\alpha \)-integrable on \([a,b]\). Then the function \( F : [a,b] \to \mathbb{R} \), defined as \( F(a) = 0 \) and \( F(x) = \int_{a}^{x} f \) for \( x \in (a,b) \), is called the primitive of \( f \) on \([a,b]\).

Recall that a function \( F : [a,b] \to \mathbb{R} \) is said to be of bounded variation on \([a,b]\) if

\[
V(F,[a,b]) = \sup \left\{ \sum_{P} |F(u,v)| \right\} < \infty,
\]

where the supremum runs over all partitions \( P \) of \([a,b]\) (that is, a finite collection of non-overlapping intervals \([u,v]\) such that \( \bigcup [u,v] = [a,b] \)), and \( F(u,v) = F(v) - F(u) \).

The following theorem answers the question on exactly when \( |f| \) is \( M_\alpha \)-integrable on \([a,b]\), given that \( f \) is \( M_\alpha \)-integrable on \([a,b]\).

**Theorem 3.3.** Let a function \( f : [a,b] \to \mathbb{R} \) be \( M_\alpha \)-integrable on \([a,b]\) with primitive \( F \). Then \( |f| \) is \( M_\alpha \)-integrable on \([a,b]\) if and only if \( F \) is of bounded variation over \([a,b]\). In this case, we have

\[
\int_{a}^{b} |f| = V(F,[a,b]).
\]

**Proof.** Suppose that \( |f| \) is \( M_\alpha \)-integrable on \([a,b]\). Let \( P = \{ [u,v] \} \) be a partition of \([a,b]\). Then

\[
\sum_{P} |F(u,v)| = \sum_{P} \left| \int_{u}^{v} f \right| \leq \sum_{P} \int_{u}^{v} |f| = \int_{a}^{b} |f|,
\]

where the supremum runs over all partitions \( P \) of \([a,b]\).
which implies \( V(F, [a, b]) < \infty \). Thus, \( F \) is of bounded variation over \([a, b]\).

Conversely, suppose that \( F \) is of bounded variation over \([a, b]\). Let \( \epsilon > 0 \) be given. Then there exists a partition \( P = \{[x_{i-1}, x_i] : i = 1, 2, \ldots, n\} \) of \([a, b]\) such that

\[
V(F, [a, b]) - \frac{\epsilon}{2} < \sum_{i=1}^{n} |F(x_{i-1}, x_i)| \leq V(F, [a, b]).
\]

Moreover, by the \( M_\alpha \)-integrability of \( f \), there exists a gauge \( \delta_0 \) on \([a, b]\) such that

\[
|S(f, D) - F(a, b)| < \frac{\epsilon}{4}
\]

for any \( \delta_0 \)-fine \( M_\alpha \)-division \( D \) of \([a, b]\).

Define

\[
\delta(x) = \begin{cases} 
\min \{\delta_0(x), x - x_{i-1}, x_i - x\} & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \\
\min \{\delta_0(x), x - x_{i-1}, x_{i+1} - x\} & \text{if } x = x_i, 1 \leq i \leq n - 1 \\
\min \{\delta_0(x), x_1 - x\} & \text{if } x = x_0 = a \\
\min \{\delta_0(x), x - x_{n-1}\} & \text{if } x = x_n = b,
\end{cases}
\]

and let \( D = \{([u_k, v_k], \xi_k) : k = 1, 2, \ldots, m\} \) be a \( \delta \)-fine \( M_\alpha \)-division of \([a, b]\).

By the definition of \( \delta \), \( D \) has \( x_0 = a, x_1, x_2, \ldots, x_n = b \) as among its tags. Further, if \( ([u, v], \xi) \in D \), where \( \xi \in \{x_0, x_1, \ldots, x_n\} \), then \( \xi \in [u, v] \), so that these tags can be made as endpoints and

\[
V(F, [a, b]) - \frac{\epsilon}{2} < \sum_{i=1}^{n} |F(x_{i-1}, x_i)| \leq \sum_{k=1}^{m} |F(u_k, v_k)| \leq V(F, [a, b]).
\]

Thus, applying Saks-Henstock Lemma and triangle inequality, we have

\[
|S(|f|, D) - V(F, [a, b])| \leq \sum_{k=1}^{m} (|f(\xi_k)|(v_k - u_k) - |F(u_k, v_k)|) + \sum_{k=1}^{m} |F(u_k, v_k)| - V(F, [a, b])
\]

\[
< \sum_{k=1}^{m} |f(\xi_k)(v_k - u_k) - F(u_k, v_k)| + \frac{\epsilon}{2} < \epsilon.
\]

This ends the proof of the theorem. \( \square \)

**Corollary 3.4.** Let the functions \( f, g : [a, b] \to \mathbb{R} \) be \( M_\alpha \)-integrable on \([a, b]\) with \( |f(x)| \leq g(x) \) for all \( x \in [a, b] \). Then \( f \) is \( M_\alpha \)-integrable on \([a, b]\), and

\[
\left| \int_{a}^{b} f \right| \leq \int_{a}^{b} |f| \leq \int_{a}^{b} g.
\]
Cauchy extension of $M_\alpha$-integral and absolute $M_\alpha$-integrability

References


Received: March 19, 2015; Published: April 30, 2015