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Cauchy Extension of M_α-Integral and Absolute M_α-Integrability

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Abstract

Park, Ryu, and Lee recently introduced a Henstock-type integral, which lies between the Mcshane and the Henstock integrals. This paper proves the closure property of this new integral under Cauchy extension, and presents a characterization on absolute M_α-integrability.

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1 Introduction

Park, Ryu, and Lee [4] recently defined a Henstock-type integral, which they call M_α-integral. Several properties and convergence theorems of the integral were established in [1] and [4]. Most of them paralleled the usual properties of Henstock integral, including the Saks-Henstock Lemma [4, Lemma 2.5]. Moreover, by providing examples, it was also shown in [4] that M_α-integral lies strictly between the McShane integral (which is known to be equivalent to the Lebesgue integral (see [2])) and the Henstock integral.

Let $\alpha > 0$ be a constant, and $I = [a, b]$ a non-degenerate closed and bounded interval in $\mathbb{R}$.

(1) A *partial division* D of I is a finite collection of interval-point pairs $([u, v], \xi)$ such that the closed intervals $[u, v]$ are non-overlapping, $\bigcup[u, v] \subseteq I$, and $\xi \in I$. If $\bigcup[u, v] = I$, we call the division D simply a *division*.

(2) A positive function defined on I is called a *gauge* on I.

(3) Let δ be a gauge on I, and $D = \{([u, v], \xi)\}$ a partial division of I. If $[u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi))$ for all $([u, v], \xi) \in D$, then we say that D is a *δ-fine McShane partial division*. Moreover, if D is a McShane partial division such that $\xi \in [u, v]$ for all $([u, v], \xi) \in D$, then D is called a *δ-fine Henstock partial division*.

(4) A McShane division $D = \{([u, v], \xi)\}$ of I is said to be an *M_α-division* if

$$\sum_{([u,v],\xi)\in D} \mathrm{dist}(\xi, [u, v]) < \alpha,$$

where $\mathrm{dist}(x, J) = \inf\{|y - x| : y \in J\}$.

(5) Let $D = \{([u, v], \xi)\}$ be a partial division on I, and f a real-valued function defined on I. We write

$$S(f, D) = \sum_{([u,v],\xi)\in D} f(\xi)(v - u).$$

With these terms and notations, the definition of M_α-integrability can now be presented.

Definition 1.1 ([4, Definition 2.1])**.** A function $f : [a, b] \to \mathbb{R}$ is *M_α-integrable* if there exists a real number A such that, for each $\epsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$|S(f, D) - A| < \epsilon$$

for each δ-fine M_α-division D of $[a, b]$. Here, A is called the *M_α-integral* of f on $[a, b]$, and we write $A = \int_a^b f$.

Moreover, f is *M_α-integrable on* $E \subseteq [a, b]$ if $f\chi_E$ is M_α-integrable on $[a, b]$, where χ_E is the characteristic function over E, and we write $\int_a^b f\chi_E = \int_E f$.

This paper proves the closure property of this new integral under Cauchy extension (Theorem 2.1) and a characterization on absolute M_α-integrability (Theorem 3.3).

2 Cauchy Extension

The following theorem shows that M_α-integral is closed under Cauchy extension, a property that is not valid for McShane integral.

Theorem 2.1 (Cauchy Extension). *Let $f : [a, b] \to \mathbb{R}$ be M_α-integrable on $[c, b]$ for each $c \in (a, b)$. Then f is M_α-integrable on $[a, b]$ if and only if $\lim_{c\to a^+} \int_c^b f$ exists. In this case, we have*

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f.$$

Proof. Suppose that f is M_α-integrable on $[a, b]$, and let $\epsilon > 0$ be given. Then there exists a gauge δ on $[a, b]$ such that if D is a δ-fine M_α-division of $[a, b]$, then

$$\left| S(f, D) - \int_a^b f \right| < \epsilon.$$

Moreover, for each $c \in (a, b)$, there exists a gauge δ_c on $[c, b]$ such that if D_c is a δ_c-fine M_α-division of $[c, b]$, then

$$\left| S(f, D_c) - \int_c^b f \right| < \epsilon.$$

For each $c \in (a, b)$, we may assume that $\delta_c(x) \le \delta(x)$ for all $x \in [c, b]$.

Choose $s \in (a, a + \delta(a))$ such that $|f(a)|(s - a) < \epsilon$. Let D_s be a δ_s-fine M_α-division of $[s, b]$. Then $D = D_s \cup \{([a, s], a)\}$ is a δ-fine M_α-division of $[a, b]$. Hence, we have

$$\left| \int_a^b f - \int_s^b f \right| \le \left| \int_a^b f - S(f, D) \right| + \left| S(f, D_s) - \int_s^b f \right| + |f(a)|(s - a) < 3\epsilon.$$

Since s is chosen arbitrarily close to a, we obtain $\lim_{c\to a^+} \int_c^b f$ exists and is equal to $\int_a^b f$.

Conversely, suppose that $\lim_{c\to a^+} \int_c^b f = A$ exists, and let $\{x_0, x_1, x_2, \ldots\}$ be a strictly decreasing sequence of real numbers such that $x_0 = b$ and $x_n \to a$. Then f is M_α-integrable on $[x_n, x_{n-1}]$ for every $n \in \mathbb{Z}^+$.

Let $\epsilon > 0$ be given. For each integer $n \ge 1$, choose a gauge δ_n on $[x_n, x_{n-1}]$ such that if D_n is a δ_n-fine M_α-division of $[x_n, x_{n-1}]$, then

$$\left| S(f, D_n) - \int_{x_n}^{x_{n-1}} f \right| < \frac{\epsilon}{2^n}.$$

Furthermore, choose an integer $N > 0$ such that

$$\left| \int_t^b f - A \right| < \epsilon \qquad \text{and} \qquad |f(a)|(t - a) < \epsilon$$

for every $t \in (a, x_N]$.

Define a gauge δ on $[a,b]$ as follows:

$$\delta(x) = \begin{cases} x_N - x & \text{if } x = a \\ \min\{\delta_1(x), x - x_1\} & \text{if } x \in (x_1, x_0] \\ \min\{\delta_n(x), x - x_n, x_{n-1} - x\} & \text{if } x \in (x_n, x_{n-1}),\ n \geq 2 \\ \min\{\delta_n(x), \delta_{n+1}(x), x_{n-1} - x, x - x_{n+1}\} & \text{if } x = x_n,\ n \geq 1. \end{cases}$$

Let D be a δ-fine M_α-division of $[a,b]$. Then

$$D = \{([a,t], a)\} \cup D_t \cup \bigcup_{i=1}^{m} D_i,$$

where $t \in (x_{m+1}, x_m]$ for some integer $m \geq N$, D_t a δ_{m+1}-fine M_α-division of $[t, x_m] \subset (x_{m+1}, x_m]$, and D_i a δ_i-fine M_α-division of $[x_i, x_{i-1}]$. Thus, by Saks-Henstock Lemma, we have

$$\begin{aligned} |S(f,D) - A| &\leq |f(a)|(t-a) + \left| S(f, D_t) - \int_t^{x_m} f \right| \\ &\quad + \sum_{i=1}^{m} \left| S(f, D_i) - \int_{x_i}^{x_{i-1}} f \right| + \left| \int_t^b f - A \right| \\ &< \epsilon + \frac{\epsilon}{2^{m+1}} + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} + \epsilon \\ &< 4\epsilon. \end{aligned}$$

This ends the proof of the theorem. □

Corollary 2.2. *Let $f : [a,b] \to \mathbb{R}$ be M_α-integrable on every $[c,d] \subset (a,b)$. Then f is M_α-integrable on $[a,b]$ if and only if $\lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f$ exists. In this case, we have*

$$\int_a^b f = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f.$$

3 Absolute M_α-Integrability

Recall that an integral theory is *absolute* if a function f is integrable (in the sense of that theory) if and only if $|f|$ is also integrable. Both Riemann and McShane integrals are absolute (see [2] or [5]), but Henstock integral is known to be not (see [3]).

Example 3.1. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Define a function $f : [0,1] \to \mathbb{R}$ as follows:*

$$f(x) = \begin{cases} 2^n a_n & \text{if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right],\ n \in \mathbb{Z}^+ \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, f is M_α-integrable on $[c,1]$ for each $c \in (0,1)$. Thus, by Theorem 2.1, we have

$$\int_0^1 f = \lim_{c \to 0^+} \int_c^1 f = \sum_{n=1}^{\infty} a_n.$$

However, it is not difficult to see that $|f|$ would be M_α-integrable if and only if the given series is absolutely convergent.

In general, due to the closure under Cauchy extension, it is possible for a function f to be M_α-integrable on $[a,b]$, but $|f|$ is not. We have the following proposition.

Proposition 3.2. *The M_α-integral is not absolute.*

Let a function $f : [a,b] \to \mathbb{R}$ be M_α-integrable on $[a,b]$. Then the function $F : [a,b] \to \mathbb{R}$, defined as $F(a) = 0$ and $F(x) = \int_a^x f$ for $x \in (a,b]$, is called the *primitive* of f on $[a,b]$.

Recall that a function $F : [a,b] \to \mathbb{R}$ is said to be *of bounded variation* on $[a,b]$ if

$$V(F,[a,b]) = \sup \left\{ \sum_P |F(u,v)| \right\} < \infty,$$

where the supremum runs over all partitions P of $[a,b]$ (that is, a finite collection of non-overlapping intervals $[u,v]$ such that $\bigcup [u,v] = [a,b]$), and $F(u,v) = F(v) - F(u)$.

The following theorem answers the question on exactly when $|f|$ is M_α-integrable on $[a,b]$, given that f is M_α-integrable on $[a,b]$.

Theorem 3.3. *Let a function $f : [a,b] \to \mathbb{R}$ be M_α-integrable on $[a,b]$ with primitive F. Then $|f|$ is M_α-integrable on $[a,b]$ if and only if F is of bounded variation over $[a,b]$. In this case, we have*

$$\int_a^b |f| = V(F,[a,b]).$$

Proof. Suppose that $|f|$ is M_α-integrable on $[a,b]$. Let $P = \{[u,v]\}$ be a partition of $[a,b]$. Then

$$\sum_P |F(u,v)| = \sum_P \left| \int_u^v f \right| \le \sum_P \int_u^v |f| = \int_a^b |f|,$$

which implies $V(F,[a,b]) < \infty$. Thus, F is of bounded variation over $[a,b]$.

Conversely, suppose that F is of bounded variation over $[a,b]$. Let $\epsilon > 0$ be given. Then there exists a partition $P = \{[x_{i-1}, x_i] : i = 1, 2, \ldots, n\}$ of $[a,b]$ such that

$$V(F,[a,b]) - \frac{\epsilon}{2} < \sum_{i=1}^{n} |F(x_{i-1}, x_i)| \leq V(F,[a,b]).$$

Moreover, by the M_α-integrability of f, there exists a gauge δ_0 on $[a,b]$ such that

$$|S(f,D) - F(a,b)| < \frac{\epsilon}{4}$$

for any δ_0-fine M_α-division D of $[a,b]$.

Define

$$\delta(x) = \begin{cases} \min\{\delta_0(x), x - x_{i-1}, x_i - x\} & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \\ \min\{\delta_0(x), x - x_{i-1}, x_{i+1} - x\} & \text{if } x = x_i,\ 1 \leq i \leq n-1 \\ \min\{\delta_0(x), x_1 - x\} & \text{if } x = x_0 = a \\ \min\{\delta_0(x), x - x_{n-1}\} & \text{if } x = x_n = b, \end{cases}$$

and let $D = \{([u_k, v_k], \xi_k) : k = 1, 2, \ldots, m\}$ be a δ-fine M_α-division of $[a,b]$. By the definition of δ, D has $x_0 = a, x_1, x_2, \ldots, x_n = b$ as among its tags. Further, if $([u,v],\xi) \in D$, where $\xi \in \{x_0, x_1, \ldots, x_n\}$, then $\xi \in [u,v]$, so that these tags can be made as endpoints and

$$V(F,[a,b]) - \frac{\epsilon}{2} < \sum_{i=1}^{n} |F(x_{i-1}, x_i)| \leq \sum_{k=1}^{m} |F(u_k, v_k)| \leq V(F,[a,b]).$$

Thus, applying Saks-Henstock Lemma and triangle inequality, we have

$$\begin{aligned} \big|S(|f|,D) - V(F,[a,b])\big| &\leq \left|\sum_{k=1}^{m} (|f(\xi_k)|(v_k - u_k) - |F(u_k, v_k)|)\right| \\ &\quad + \left|\sum_{k=1}^{m} |F(u_k, v_k)| - V(F,[a,b])\right| \\ &< \sum_{k=1}^{m} |f(\xi)(v_k - u_k) - F(u_k, v_k)| + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

This ends the proof of the theorem. □

Corollary 3.4. *Let the functions $f, g : [a,b] \to \mathbb{R}$ be M_α-integrable on $[a,b]$ with $|f(x)| \leq g(x)$ for all $x \in [a,b]$. Then $|f|$ is M_α-integrable on $[a,b]$, and*

$$\left|\int_a^b f\right| \leq \int_a^b |f| \leq \int_a^b g.$$

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