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$k$-Isocoronal tilings

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**k-Isocoronal tilings**

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In this article, a framework is presented that allows the systematic derivation of planar edge-to-edge $k$-isocoronal tilings from tile-$s$-transitive tilings, $s \leq k$. A tiling $\mathcal{T}$ is $k$-isocoronal if its vertex coronae form $k$ orbits or $k$ transitivity classes under the action of its symmetry group. The vertex corona of a vertex $x$ of $\mathcal{T}$ is used to refer to the tiles that are incident to $x$. The $k$-isocoronal tilings include the vertex-$k$-transitive tilings ($k$-isogonal) and $k$-uniform tilings. In a vertex-$k$-transitive tiling, the vertices form $k$ transitivity classes under its symmetry group. If this tiling consists of regular polygons then it is $k$-uniform. This article also presents the classification of isocoronal tilings in the Euclidean plane.

1. Introduction

Historically, tilings have interested artisans as a means of expression and cultural beliefs. In recent years, scientists and mathematicians have studied tilings as mathematical models for crystal structures (both periodic and non-periodic), cell packing of viruses, algebraic codes and self-assembly of nanostructures, to name a few. Mathematicians continue to study the interesting mathematical questions posed by tiling theory using several branches of mathematics: group theory, combinatorics, graph theory, topology and dynamical systems.

One of the problems of interest in the study of tilings is the classification and characterization of tilings with transitivity properties, including the analysis of their symmetry groups. A tiling is vertex-transitive (isogonal), edge-transitive (isotoxal) or tile-transitive (isohedral) if its symmetry group forms one orbit or transitivity class of vertices, edges or tiles, respectively, in the given tiling. The Euclidean vertex-, edge- and tile-transitive tilings have been classified by Grünbaum & Shephard (1987) in their work. In a given tiling, it is possible that the vertices, edges or tiles form more than one orbit under the action of its symmetry group. Consider for instance the Euclidean tiling $\mathcal{T}^*$ shown in Fig. 1 consisting of two types of 4-gons (kites and rhombi) with symmetry group $G^* = \langle P', Q', R' \rangle \cong p6mm$ (IUCr notation) where $P'$, $Q'$, $R'$ are reflections with axes passing through the edges of the shaded triangle as shown. The vertices of $\mathcal{T}^*$ form three orbits or three transitivity classes under $G^*$ (vertices from the same transitivity class are given the same color). Moreover, three orbits of vertex coronae under the action of $G^*$ are also formed in $\mathcal{T}^*$. A vertex corona of a green vertex consists of two kites and a rhombus; three kites and three rhombi alternate around a sky-blue vertex and make up its vertex corona; and six kites constitute the vertex corona of a yellow vertex. The tiling $\mathcal{T}^*$ is called 3-isocoronal and vertex-3-transitive (3-isogonal).
An objective of this article is to contribute to the classification and characterization of tilings with transitivity properties through the study of planar edge-to-edge $k$-isocoronal tilings. The intent is to present a systematic method that will give rise to $k$-isocoronal tilings, including vertex-$k$-transitive ($k$-isogonal) tilings derived from tilings satisfying tile-transitivity properties under subgroups of their symmetry groups.

In Frettlöh & Garber (2015) tilings where all vertex coronae are congruent, called monocoronal tilings, have been explored using a combinatorial approach. However, there is not much discussion in the literature on $k$-isocoronal or $k$-isogonal tilings. There have been previous studies on Euclidean $k$-uniform tilings, but its enumeration for specific values of $k$ is far from complete. There is Kepler’s (1619) enumeration of the 1-uniform tiling, the complete enumeration of the $k$-homogeneous tilings (a set of tilings which includes the 1-uniform and 2-uniform tilings) by Krötenheerd (1969) and Chavey’s (1984) enumeration of the 3-uniform tilings. The 4-, 5- and 6-uniform tilings have been found by computer and published on the Web by Galebach (2002), but his method is not described, and no explicit proof that his lists are complete seems to be available.

Aside from the discussion of $k$-isocoronal tilings, which include vertex-$k$-transitive ($k$-isogonal) and $k$-uniform tilings, we also present in this article the derivation of isocoronal tilings in the Euclidean plane. We also provide a result that seems to be available.

The organization of this article is as follows. In Section 2, we give the definitions and notions used in the article. Section 3 contains results pertaining to $k$-isocoronal tilings. Section 4 discusses the special case when $k = 1$ and gives the list of isocoronal tilings in the Euclidean plane. Section 5 discusses tile-transitive isocoronal tilings. Section 6 presents the summary and future direction of the study.

2. General notions and definitions

A tiling $\mathcal{T}$ of $\mathbb{X}$ ($\mathbb{X}$ is either the Euclidean plane $\mathbb{E}^2$ or hyperbolic plane $\mathbb{H}^2$ in this article) is a collection of tiles $\mathcal{T} = \{t_i : i \in \mathbb{N}\}$ that is a covering $(\bigcup t_i = \mathbb{X})$ as well as a packing $[\text{Int}(t_i) \cap \text{Int}(t_j) = \emptyset \text{ if } i \neq j, \text{Int}(t) \text{ denotes the interior of tile } t]$. At times, the tiles considered are tangential polygons. A tangential polygon is a convex polygon that has an inscribed circle. A tiling is edge-to-edge if the intersection of any of its two tiles is either the common edge or common vertex of the tiles, or empty. Two tiles having a common edge are said to be adjacent.

A vertex of $\mathcal{T}$ is a point $x$ such that $x$ is a vertex of at least one tile in $\mathcal{T}$. A vertex with $w$ number of edges incident to it is said to have valence $w$. A vertex $x$ of $\mathcal{T}$ is said to be regular if the angles between each pair of consecutive edges incident to $x$ are congruent. The vertex corona of a vertex $x$ is the set of all tiles incident to $x$. Note that a vertex corona is a set of tiles, not a union of tiles.

The tile corona $C(t)$ of a tile $t$ in $\mathcal{T}$ is the set of all tiles that have non-empty intersection with $t$. Two tiles $t$ and $t'$ possibly have the same tile corona, that is $C(t) = C(t')$ but $t \neq t'$. We distinguish between these tile coronae by using the term centered tile corona $C(t)$ to mean the pair consisting of the tile corona $C(t)$ and its fixed center $t$. Throughout this article, whenever we refer to a tile corona of a tile, we mean its centered tile corona.

The symmetry group $G$ of $\mathcal{T}$ is the group of all isometries that leave $\mathcal{T}$ invariant. Consider a tile $t$ in $\mathcal{T}$ and $H \leq G$. We define the stabilizer of $t$ in $H$, denoted by $\text{Stab}_H(t)$, as the group that consists of elements of $H$ that fix $t$, that is $\text{Stab}_H(t) = \{h \in H : h \cdot t = t\}$. The set $Ht = \{ht \in T : h \in H\}$ is the orbit of $t$ under the action of $H$.

A tiling is $k$-isocoronal if its vertex coronae belong to $k$ orbits or $k$ transitivity classes under the action of its symmetry group. A 1-isocoronal tiling is simply referred to as isocoronal. A $k$-isocoronal tiling is said to be of type $\{p_1, p_2, \ldots, p_{r_1}; \ldots; p_{k1}, p_{k2}, \ldots, p_{r_k}\}$, where a vertex corona in the $i$th transitivity class consists of $r_i$ number of polygons $p_{i1}, p_{i2}, \ldots, p_{i_r}$-gons (in cyclic order). A type of $k$-isocoronal tiling is divided into families of isocoronal tilings. A family of $k$-isocoronal tilings is distinguished by the symmetry group of the tiling and the symmetry group of the tiles appearing in the vertex corona. For example, the tiling $T^+$ in Fig. 1 is of type $(4^3; 4^6; 4^8)$, where the vertex corona in the first, second and third transitivity class, respectively, consists of 3, 6 and number of 4-gons. This 3-isocoronal tiling belongs to the family $(4^{(2)}; 4^{(2)}; 4^{(2)})\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}; 4^{(2)}\cdot 4^{(2)}\cdot 4^{(2)}$. A 4-gon of symmetry group $D_4$.
(dihedral group of order 2) and $D_2$ (dihedral group of order 4) is denoted by $4^{(0)}$ and $4^{(0)}$, respectively, where the symmetry group of the 4-gon is denoted as a superscript. Every $k$-isocoronal tiling is vertex-$k$-transitive or $k$-isogonal, that is, its vertices belong to $k$ orbits or $k$ transitivity classes under its symmetry group.

A tiling is edge-$k$-transitive or $k$-isotoxal if its edges belong to $k$ orbits under its symmetry group. The tiling in Fig. 1 is edge-2-transitive. The two sets of edges colored differently form two transitivity classes under the action of $G^*$. In this section we present two results that facilitate the construction of a $k$-isocoronal tiling. The second result particularly gives a $k$-isocoronal tiling consisting of regular polygons.

**Theorem 3.1.** Let $T$ be an edge-to-edge tiling of $\mathbb{X}$ with symmetry group $G$. Consider a subgroup $H$ of $G$. Suppose $H$ forms $k$ orbits of tiles in $T$. Then there exists an edge-to-edge $s$-isocoronal tiling $T^*$, $s \leq k$, with symmetry group $G^* \geq H$. Moreover, if $G^* = H$ then $T^*$ is $k$-isocoronal.

**Proof.** Let $T$ be a tiling with symmetry group $G$. Let $H$ be a subgroup of $G$. Suppose $H$ forms $k$ orbits of tiles in $T$, that is, $T = H\Gamma_1 \cup H\Gamma_2 \cup \ldots \cup H\Gamma_k$ where $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are tiles in $T$ representing the tile orbits under $H$.

Consider a point $x_i \in \text{Int}(\Gamma_i), i = 1, 2, \ldots, k$, such that $h\Gamma_i = \Gamma_j$ for every $h \in \text{Stab}_G(\Gamma_i)$. Form $\mathcal{P} = H\Gamma_1 \cup H\Gamma_2 \cup \ldots \cup H\Gamma_k$. This will result in every tile in $T$ containing one point from $\mathcal{P}$. Connect by an edge two points in $\mathcal{P}$ that belong to two adjacent tiles in $T$. This results in an edge-to-edge tiling $T^*$ with polygons as tiles and vertices consisting of points from $\mathcal{P}$.

Suppose $\Gamma_i$ has $r_i$ vertices with valences $v_{11}, v_{12}, \ldots, v_{1r_i}$, then the vertex corona $A_i$ of $x_i \in \text{Int}(\Gamma_i)$, consists of $v_{11}, v_{12}, \ldots, v_{1r_i}$-gons. Moreover, assuming $\Gamma_i$ has $r$ vertices with valences $v_{j1}, v_{j2}, \ldots, v_{jr}$, then the vertex corona $A_i$ consists of $v_{j1}, v_{j2}, \ldots, v_{jr}$-gons. Consider two vertices $x, y$ in $T^*$ such that $x, y \in H\Gamma_i$. Then $x \in t, y \in t'$, where $t, t' \in H\Gamma_i$. Thus, there exists $h \in H$ such that $t' = ht$. Consequently, $C(t') = h(C(t))$ where $C(t), C(t')$ is the centered tile corona of $t, t'$, respectively. This implies $B = hA$, where $A, B$ is the vertex corona of $x, y$, respectively. Hence the vertex corona of any two vertices in $H\Gamma_i$ are congruent. The vertex corona of these vertices will consist of $v_{11}, v_{12}, \ldots, v_{1r_i}$-gons. Using similar arguments, the vertex corona of any two vertices in $H\Gamma_j, j = 2, 3, \ldots, k$, are congruent, and will consist of $v_{j1}, v_{j2}, \ldots, v_{jr}$-gons.

Now, take two vertices $v, w$ in $T^*$ such that $v \in H\Gamma_i$ and $w \in H\Gamma_j$, $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$. Consider the respective vertex corona $A_i, A_j$ of $v, w$. Then $hA_i \neq A_j$ for all $h \in H$ since $t_i \notin H\Gamma_j$.

For the tiling $T^*$, we have $HT^* = T^*$. Thus, the symmetry group $G^*$ of $T^*$ contains $H$. If $G^* = H$, then there are $k$ transitivity classes of vertex coronae under $G^*$, so that $T^*$ is $k$-isocoronal. Suppose $G^* > H$. If $g^*A_i = A_j$ for some $g^* \in G^* - H$, $i, j \in \{1, 2, \ldots, k\}, i \neq j$, then there are $s$ transitivity classes of vertex coronae under $G^*$, where $s < k$ and $T^*$ is an $s$-isocoronal tiling. Otherwise, $T^*$ is a $k$-isocoronal tiling.

In the above result the resulting $s$-isocoronal tiling $T^*$ is dual to the tiling $T$ in the sense that there exists an incidence reversing bijection between the set of vertices, edges and

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**Figure 2**

(a)–(c) Construction of a 3-isocoronal tiling ($4^3; 4^6; 4^2$) from the [4$^3; 4^6$] tiling; (d) a 2-isocoronal tiling ($4^2; 4^6$) from the [4$^3; 4^6$] tiling.
tiles of $T$, and respectively, the set of tiles, edges and vertices of $T^*$.

Let us illustrate the application of Theorem 3.1 by discussing the construction of the 3-isocoronal tiling $T^*$ given in Fig. 1. We start with the tiling $T := [4; 4^6]$ with symmetry group $G = (P, Q, R, S) \cong p6mm$ where $P, Q$ and $R$ are reflections with axes shown in Fig. 2(a). Consider $H = (P, Q, RQPQR) \cong p6mm$ an index-3 subgroup of $G$. $H$ forms three orbits of tiles in $T$: one orbit consists of equilateral triangles, another orbit consists of blue regular hexagons and the third orbit consists of pink regular hexagons. We have $T = Ht_1 \cup Ht_2 \cup Ht_3$ where $t_1$ is a triangle, $t_2$ a blue hexagon and $t_3$ a pink hexagon as labeled in Fig. 2(a).

Consider $x_i \in \text{Int}(t_i)$, $x_2 \in \text{Int}(t_2)$ and $x_3 \in \text{Int}(t_3)$ such that $x_i$ lies on the axis of $Q$, and $x_2, x_3$ are the respective incenters of $t_2$ and $t_3$ [Fig. 2(b)]. The position of $x_i, i = 1, 2, 3$, is chosen such that $hx_i = x_i$ for all $h \in \text{Stab}_H(t_i)$ where $\text{Stab}_H(t_i) = \langle Q \rangle \cong D_3$, $\text{Stab}_H(t_2) = \langle P \rangle \cong D_6$ and $\text{Stab}_H(t_3) = \langle P, Q, RQPQR \rangle \cong D_5$. We form $\mathcal{P} = Hx_1 \cup Hx_2 \cup Hx_3$ [Fig. 2(b)]. Connecting by an edge the points in $\mathcal{P}$ that lie in the interiors of adjacent tiles results in the tiling $T^*$ of type $[4^6; 4^6; 4^6]$ given in Fig. 2(c). As shown in Fig. 2(b), $t_1$ has three vertices, each of valence 4, so the vertex corona $A_i$ of $x_i$ consists of three 4-gons (two kites and a rhombus). The vertex corona $A_2$ of $x_2$ consists of six 4-gons (kites) since each of the vertices of $t_2$ has valence 4. The vertex corona $A_3$ of $x_3$ also consists of six 4-gons (three kites and three rhombi arranged alternately) since $t_3$ has six vertices, each of which has valence 4. The vertex corona of a vertex $hx_i \in Hx_i$ of $T^*$ is congruent to $A_i, i = 1, 2, 3$, so there are three orbits of vertex coronae under $H$. The symmetry group $G^*$ of $T^*$ is $G^* = H = (P, Q, RQPQR) \cong p6mm$.

We remark that the tiles comprising the vertex corona that are formed in the resulting tiling vary depending on the choices of $x_1, x_2, x_3$ and how these points are positioned in the respective interiors of $t_1, t_2, t_3$. Suppose, for instance, we take a point $x_i^*$ as the incenter of $t_i$ and consider the points $x_j$ and $x_k$ in the same manner as described earlier. We obtain the respective vertex coronae $A_{i}^*$, $A_{j}^*$ and $A_{k}^*$ of $x_i^*, x_j, x_k$ consisting of three, six and six rhombi, respectively. The points $x_i^*, x_j, x_k$ form a triangle symmetric under the reflection $R \in G$, and we obtain $A_{i}^* = RA_{j}^* = RA_{k}^*$ as shown in Fig. 2(d). The resulting tiling obtained from $T$ is a 2-isocoronal tiling $T^*_2$ of type $[4^6; 4^6]$ [Fig. 2(d)]. In this case, the symmetry group of $T^*_2$ is $G_2^* = G = H \cup RH \cup (RQPQR)H$.

As the next example, we present the construction of a 4-isocoronal tiling $T^*$ of type $(4.8.10; 4.8.10; 4.8.10; 4.8.10)$ of the hyperbolic plane $\mathbb{H}^2$. Consider the tile-transitive tiling $[4.8.10]$ of $\mathbb{H}^2$ with symmetry group $G = (P, Q, R) \cong \ast 542$ where $P, Q, R$ are reflections with axes shown in Fig. 3(a). Its tiles are 3-gons with angles $\pi/5, \pi/4, \pi/2$. Now, let $H = \langle QP, RQPQR \rangle \cong 552$ where $[G : H] = 4$. The generators $QP$ and $RQPQ$ of $H$ are fivefold rotations with pentagonal centers shown in Fig. 3(b). The group $H$ forms four transitivity classes of tiles in the [4.8.10] tiling [tiles belonging to the same orbit are given the same color in Fig. 3(a)]. Consider a point $x_i \in \text{Int}(t_i), i = 1, 2, 3, 4$, for a triangle tile $t_i$ where $\text{Stab}_H(t_i) = \{e\}$, and form $\mathcal{P} = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4$ [Fig. 3(b)]. Connecting the points in $\mathcal{P}$ yields the 4-isocoronal tiling $T^*$ in Fig. 3(c), edges of the same length in $T^*$ are given the same color. The vertex coronae of $x_1, x_2, x_3, x_4$ are pairwise non-congruent; each vertex corona consists of a different set of 4-gons, 8-gons and 10-gons. The symmetry group of $T^*$ is $H \cong 552$. This tiling belongs to the family $(4(C_1)8(C_2)10(C_3), 4(C_1)8(C_2)10(C_3),$ $...).$
There are two non-congruent 10-gons with symmetry group \(C_5\) (cyclic group of order 5) which are distinguished using subscripts.

Note that the hyperbolic symmetry groups in the article are expressed in Conway’s orbifold notation. The orbifold notation is based on the type of symmetries occurring in the group. The symbol \(\ast\) denotes a reflection, \(\times\) a glide reflection, \(\circ\) a translation and a positive integer \(s\) indicates an \(s\)-fold rotation. If \(s\) comes after \(\ast\), the center of the corresponding rotation lies on the axis of reflection, so the symmetry there is dihedral of order \(2s\). A center of rotation of order \(n\), \(n > 2\), will be labeled by an \(n\)-gon. A twofold rotation will be labeled by an oval. The axes of reflections and glide reflections will be indicated by solid lines and broken lines, respectively.

**Theorem 3.2.** An edge-to-edge \(k\)-isocongral tiling \((v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk})\) of \(X\) by regular polygons can be obtained by the process of Theorem 3.1 from an edge-to-edge tile-\(k\)-transitive tiling \([v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk}]\) by tangential polygons with regular vertices, and conversely. Both tilings have the same symmetry group.

**Proof.** Let \(T = [v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk}]\) be an edge-to-edge tile-\(k\)-transitive tiling with symmetry group \(G\). Since \(T\) is tile-\(k\)-transitive, \(G\) forms \(k\) orbits of tiles in \(T\), that is, \(T = Gt_1 \cup Gt_2 \cup \ldots \cup Gt_k\) where \(t_1, t_2, \ldots, t_k\) are tiles in \(T\) representing the tile orbits under \(G\). Consider the incenters \(x_i\) of \(t_i\), \(i = 1, 2, \ldots, k\), and form \(P = Gt_1 \cup Gt_2 \cup \ldots \cup Gt_k\). Every tile in \(T\) contains a point from \(P\) that is the center of the incircle of the tile. Connecting by an edge two points from \(P\) that belong to two adjacent tiles in \(T\), we obtain an edge-to-edge \(k\)-isocongral tiling \(T^* = (v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk})\) with polygons as tiles and vertices consisting of points from \(P\) [Fig. 4(a)].

Consider \(x_i \in \text{Int}(t_i)\), the incenter of \(t_i\) where \(t_i\) has \(r_i\) vertices with valences \(v_{i1}, v_{i2}, \ldots, v_{ik}\) (in cyclic order). Given a vertex \(z\) of \(t_i\) with valence \(v_{11}\), a \(v_{11}\)-gon is formed (with \(z\) in its interior) by connecting the incenters of tiles incident to \(z\) [Fig. 4(b)]. Since \(z\) is a regular vertex, the incenters of two adjacent tiles that are incident to \(z\) intersect at the same point on the common edge of the tiles. Every edge of the \(v_{11}\)-gon has the same length that is twice the radius of the incircles. Moreover, the vertex angles of the \(v_{11}\)-gon are congruent since the incentres are congruent. Thus the \(v_{11}\)-gon is regular. For every other vertex of \(t_i\) with valence \(v_{1j}, j = 2, \ldots, r_i\), a \(v_{1j}\)-gon is also formed. Thus, regular \(v_{11}, v_{12}, \ldots, v_{1r_i}\)-gons are formed incident to \(x_i\) (incenter of \(t_i\)), which constitute the vertex corona of \(x_i\).

Consider two vertices \(x, y\) in \(T^*\) such that \(x, y \in Gx_j\) [Fig. 4(a)]. Then \(x \in t, y \in t'\) where \(t, t' \in Gt_i\). Hence there exists \(g \in G\) such that \(t' = gt\). Then \(C(t) = C(t') = C(0)\) where \(C(0)\) is the centered tile corona of \(t, t'\), respectively. This implies \(B = gA\), where \(A, B\) is the vertex corona of \(x, y\), respectively. Thus the vertex coronae of any two vertices in \(Gx_j\) are congruent. Similarly, it can be shown that the vertex coronae of any two vertices in \(Gx_j\) are congruent.

Now, take two vertices \(v, w\) in \(T^*\) such that \(v \in Gx_j\) and \(w \in Gx_k\), \(i, j \in \{1, 2, \ldots, k\}, i \neq j\) [Fig. 4(a)]. Consider the respective vertex coronae \(A_i, A_j\) of \(v, w\). Then \(gA_i \neq A_j\) for all \(g \in G\) since \(t_i \notin Gt_k\). Thus there are \(k\) orbits or \(k\) transitivity classes of vertex coronae under \(G\).

For the tiling \(T^*\), we have \(GT^* = T^*\) so that \(G\) is the symmetry group of \(T^*\). Therefore \((v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk})\) is an \(k\)-isocongral tiling by regular polygons.

Conversely, suppose \(T^* = (v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk})\) is an edge-to-edge \(k\)-isocongral tiling by regular polygons with symmetry group \(G^*\). Connect using an edge the incenters belonging to two adjacent tiles in \(T^*\). An edge-to-edge tiling \(T^*\) is formed with polygons as tiles, and the incenters of the tiles in \(T^*\) as vertices [Fig. 5(a)].

Consider a vertex \(z_i^*\) of \(T^*\) with a vertex corona consisting of regular \(v_{11}, v_{12}, \ldots, v_{1k}\)-gons. There are \(r_i\) polygons incident to \(z_i\), so connecting the incenters of these polygons will form an \(r_i\)-gon \(P_i \in T^* = (v_{11}, v_{12}, \ldots, v_{1k}; \ldots; v_{k1}, v_{k2}, \ldots, v_{kk})\). Since the vertex corona of \(z_i^*\) consists of regular \(v_{11}, v_{12}, \ldots, v_{1k}\)-gons, then vertices of \(P_i\) will have valences \(v_{11}, v_{12}, \ldots, v_{1k}\). The vertices of \(P_i\) are regular by the following arguments. Note that the incentres of two adjacent regular polygons in \(T^*\) intersect at the midpoints of their common edges. Thus, the segments connecting the incenters of the adjacent polygons are perpendicular bisectors of their common edges. It follows that the angles formed by

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**Figure 4**

(a) Tiling \(T^*\) (blue) superimposed on tiling \(T\) (gray); the vertices \(v, w, x, y\) of \(T^*\) are shown. (b) Construction of a regular 3-gon (with blue edges) from tangential tiles of \(T\) meeting at a regular vertex \(z\) with valence 3.
then the vertices (incenters) of $P_1$ are regular [Fig. 5(b)].

$P_1$ is tangential. The sides of any given angle of $P_1$ pass through the midpoints of the sides of a particular regular polygon incident to $z_1^*$. So the angle bisector of the angle passes through $z_1^*$. Thus, $P_1$ has an incenter, which is $z_1^*$ [Fig. 5(b)].

In a similar manner, given a vertex $z_i^*$ of $T^*$ with a vertex corona consisting of regular $v_{j1^*}, v_{j2^*}, \ldots, v_{jk^*}$-gons, $j = 2, \ldots, k$, we obtain an $r_j$-gon $P_j$ that is tangential with regular vertices. Thus $T$ is a tiling by $r_1^*, r_2^*, \ldots, r_k^*$-gons that are tangential with regular vertices.

Take two $r_j$-gons $t_i, t_j$ in $T$ for some $i \in \{1, 2, \ldots, k\}$ such that $x^* \in \operatorname{Int}(t_i), y^* \in \operatorname{Int}(t_j)$ are vertices of $T^*$ with respective vertex coronae $A_i, A_j$ (both consisting of regular $v_{i1^*}, v_{i2^*}, \ldots, v_{ik^*}$-gons) where $A_j = g^*A_i$ for some $g^* \in G^*$ [Fig. 5(a)]. Then $t_i = g^*t_j$ and $t_i, t_j$ belong to the same $G^*$ orbit of tiles.

Now take two tiles $r_j$-gon $t$ and $r_j$-gon $t'$ in $T$ such that $x^* \in \operatorname{Int}(t), y^* \in \operatorname{Int}(t')$ are vertices of $T^*$ with respective vertex coronae $A_i$ and $A_j$ where $A_i \neq g^*A_j$ for all $g^* \in G^*$ [Fig. 5(a)]. Then $t' \neq g^*t$ for all $g^* \in G^*$. This implies that there are $k$ orbits of tiles in $T$ under $G^*$. The symmetry group of $T$ is also $G^*$. Thus $T$ is a tile-$k$-transitive tiling $[v_{i1^*}, v_{i2^*}, \ldots, v_{ik^*}, \ldots; v_{k1^*}, v_{k2^*}, \ldots, v_{kk^*}]$. □

We now present the construction of a Euclidean 8-isocoronal tiling by regular polygons (8-uniform tiling). Consider the tile-$8$-transitive tiling $[3^6; 3^6, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4; 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4]$ by tangential polygons with regular vertices shown in Fig. 6(a) where the tiles belonging to one orbit are given the same color. The 8-uniform tiling is given in Fig. 6(b), obtained by connecting the incenters of the tiles in the tile-$8$-transitive tiling. The composition of polygons (in cyclic order) in the vertex coronae of the 8-uniform tiling are as follows: (i) six 3-gons (red); (ii) another six 3-gons (blue); (iii) four 3-gons and a 6-gon (yellow); (iv) three 3-gons and two 4-gons (light blue); (v) two 3-gons, a 4-gon, a 3-gon and a 4-gon (green); (vi) another two 3-gons, a 4-gon, a 3-gon and a 4-gon (cyan); (vii) a 3-gon, two 4-gons and a 6-gon (orange); (viii) a 6-gon, a 3-gon, a 6-gon and a 3-gon (pink). The 8-uniform tiling and the tile-$8$-transitive tiling both have symmetry group $G \cong p6mm$. The 8-uniform tiling belongs to the family

![Figure 5](image1.png)

**Figure 5**
(a) Tiling $T$ (red) superimposed on tiling $T^*$ (gray); the tiles $t_1, t_2, t_1, t_1$ in $T$ are shown. (b) Construction of a regular 6-gon $P_1$ (with red edges) from regular polygons incident to vertex $z_1^*$ of $T$. $P_1$ is tangential with the incenter at $z_1^*$.

![Figure 6](image2.png)

**Figure 6**
(a) Tile-$8$-transitive tiling $[3^6; 3^6, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4; 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4]$; (b) 8-isocoronal tiling $[3^6; 3^6, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4; 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4, 3^4]$ by regular polygons.
A second example is the 3-uniform tiling \(3.5^3; 3.5^3; 3.5^3\) of \(\mathbb{H}^2\) [Fig. 7(b)] constructed from a tile-3-transitive tiling \([3.5^3; 3.5^3; 3.5^3]\) given in Fig. 7(a) (tiles belonging to one orbit are given the same color). A vertex corona of the 3-uniform tiling consists of three 5-gons and a 3-gon and is presented in Fig. 7(b) where three vertex coronae belonging to three different transitivity classes are also shown. The 3-uniform tiling and the tile-3-transitive tiling have symmetry group \(G \cong 32\ast o\). 

4. Isocoronal tilings of \(\mathbb{E}^2\)

The next theorems give the isocoronal tilings of \(\mathbb{E}^2\). These results arise by applying Theorems 3.1 and 3.2 when \(k = 1\).

**Theorem 4.1.** In \(\mathbb{E}^2\), there exist exactly eighty-seven (87) edge-to-edge isocoronal tilings by convex polygons obtained from edge-to-edge tile-transitive tilings by convex polygons. From these tilings, twenty (20) are tile-transitive (Fig. 8).

A tiling in Fig. 8 is labeled according to the tiles in its vertex corona. The tiles are described as follows. A \(p\)-gon with \(C_d\) symmetry \((C_d = \text{cyclic group of order } d)\) is denoted by \(p^{(d)}\). If a \(p\)-gon has \(D_a\) symmetry, it is denoted by \(p^{(2a)}\). Two non-congruent \(p\)-gons with the same symmetry are distinguished using subscripts. Example: two non-congruent regular 4-gons are denoted by \(4\) and \(4_a\). The symmetry group of the tilings presented in Figs. 8 and 9 are described in Conway’s orbifold notation. The correspondence between Conway notation and IUCr notation of symmetry groups is as follows: 2222 (2p), 333 (p3), 442 (p4), 632 (p6), *2222 (p2mm), *333 (p3m1), *442 (p4mm), *632 (p6mm), 2*22 (c2mm), 3*3 (p31m), 4*2 (p4gm), 22* (p2mg) and 22× (p2gg).

Each of the 87 tilings in the list is representative of a family of isocoronal tilings of a particular symmetry group with a given set of polygons and their respective symmetry-group types appearing in the vertex corona. For example, for the case with a 4-gon, a 6-gon and a 12-gon in the vertex corona [an isocoronal tiling of type (4.6.12)], there are five families of tilings in the list (Nos. 47–51), each representing a different combination of symmetry-group types of a 4-gon, a 6-gon and a 12-gon in the vertex corona. Compare for instance the family of tilings No. 48 \((4^{(4)}); 6^{(12)}; 12^{(12)}\) with the family of tilings No. 49 \((4^{(4)}); 6^{(12)}; 12^{(12)}\) where the 6-gons in the former and the latter have \(D_3\) and \(D_6\) symmetry groups, respectively. Note further that the families of tilings No. 23 and No. 25 have the same label \((4^{(4)}); 4^{(4)}; 4^{(4)}\) but both appear in the list since the tilings have different symmetry groups.

To arrive at the list of isocoronal tilings we apply Theorem 3.1 where \(k = 1\). We consider the tile-transitive tilings by convex polygons listed in Schattschneider & Dolbilin (1998). Given a tile-transitive tiling \(T\), we consider a subgroup \(H\) of its symmetry group \(G\) such that \(H\) acts transitively on the tiles, that is, \(Ht = T\) where \(t\) is a tile in \(T\). The computer software Groups, Algorithms and Programs (GAP) (The GAP Group, 2015) was employed in generating the list of subgroups, of finite index, of the symmetry groups of the tilings. Given a subgroup \(H\) of \(G\) we construct an isocoronal tiling from \(T\) by (i) considering a point \(x \in \text{Int}(t)\) such that \(hx = x\) for all \(h \in \text{Stab}_T(t)\), (ii) taking the orbit \(Hx\) of \(x\) under \(H\) and (iii) connecting the points that belong to adjacent tiles. Details on the derivation of the tilings are given in Taganap (2017).

Elaborating on (i), we have the following. For the case where \(\text{Stab}_T(t)\) contains a rotational symmetry, we choose \(x\) to be the center of rotation. If \(\text{Stab}_T(t)\) does not have a rotational symmetry but has a reflection symmetry, we pick \(x\) to be a point in the axis of reflection symmetry. We pick \(x\) to be (a) a center or incenter of a tile, and (b) any other point in the axis of reflection symmetry. For (b), there are many ways one can choose this point in the line of symmetry; choosing one point...
in the axis is sufficient to give rise to a representative tiling in the list. If we take for example two distinct points on the line, then the two tilings obtained belong to one family up to similarity of tiles.

Now, for the case where \( \text{Stab}_H(t) = \{e\} \), any point in the interior of tile \( t \) satisfies \( hx = x \) for all \( h \in \text{Stab}_H(t) \). However, we select the possible positions of \( x \in \text{Int}(t) \) that will allow us to generate a complete representation of isocoronal tilings from the given \( T \) with different symmetry groups (we look at symmetry group \( H' \) such that \( H \leq H' \leq G \)) and with vertex coronae consisting of polygons with varying symmetry-group types. First, we take \( x \) as the incenter (for tangential polygons with regular vertices). Then we get a tiling with a vertex corona consisting of regular polygons. This can be referred to as the case where an \( n \)-gon in the vertex corona has symmetry group \( D_n \). It is possible that an \( n \)-gon in the vertex corona has as symmetry group a non-identity subgroup \( S \) of \( D_n \). We exhaust all the possibilities for the positions of \( x \) for the resulting \( n \)-gon to have varying \( S \) and this is obtainable by taking \( x \) as follows: (a) \( x \) is a center of rotation symmetry or \( x \) lies on an axis of reflection symmetry belonging to \( \text{Stab}_G(t) \); (b) \( x \) lies on an angle bisector of tile \( t \); and (c) \( x \) is a point besides that described in (a)–(b). As explained above, choosing one point lying on an axis of reflection, or on a given angle bisector is sufficient to give rise to a representative tiling in the list with tiles up to similarity. Whichever way we take the point described in (c), we get a tiling belonging to one family of tilings in the list.

For example, let \( T \) be the \([4.6.12]\) tiling by triangles with angles \( \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6} \), which has symmetry group \( G = \langle P, Q, R \rangle \cong *632 \). The symmetry group \( G \) forms one orbit of tiles in \( T \) and is the only suitable choice for \( H \) for this tiling. We have \( \text{Stab}_G(t) = \{e\} \). The possible positions of \( x \in \text{Int}(t) \) to be considered are the incenter of \( t \), a point lying on an angle bisector of \( t \) (leading to three possibilities for every angle bisector of the 3-gon), and a point that is neither the incenter nor is a point lying on an angle bisector of \( t \).

Choosing \( x \) as the incenter, we obtain the family of isocoronal tiling No. 47 \((4^{D_4}6^{D_6}12^{D_{12}})\) with a vertex corona consisting of regular polygons. If \( x \) lies on an angle bisector we get three possible isocoronal tilings, each belonging to the families No. 49 \((4^{D_2}6^{D_6}12^{D_{12}})\), No. 50 \((4^{D_4}6^{D_3}12^{D_{12}})\) and No. 51.
whenever we take \( x \) to be neither the incenter nor lying on an angle bisector, we obtain a tiling belonging to the family No. 48 \((4^{([p, q, r])}; 6^{([p, q, r])}; 12^{([p, q, r])})\). All the symmetry groups of these tilings are \#632.

To illustrate the method further, we consider \( T_1 := [3, 12, 2] \) a tiling by isosceles triangles with symmetry group \( G_1 = (P, Q, R) \cong 632 \). Two subgroups \( H_1, H_2 \) of \( G_1 \) satisfy the condition that \( H_1 \) or \( H_2 \) forms one orbit of tiles in \( T \). We first consider \( H_1 = (P, QR) \cong 3 \times 3 \), \( [G : H_1] = 2 \). For a tile \( t \in T_1 \), we have \( \text{Stab}_{H_1}(t) \cong [e] \) and \( \text{Stab}_{G_1}(t) \cong D_1 \). Taking \( x \) as the incenter, we obtain the family of tilings No. 52 \((3^{([p, q, r])}; 12^{([p, q, r])}; 12^{([p, q, r])})\) with symmetry group \#632; taking \( x \) not on the axis of reflection symmetry belonging to \( \text{Stab}_{G_1}(t) \) (which is also an angle bisector in this case) yields a tiling belonging to the family No. 53 \((3^{([p, q, r])}; 12^{([p, q, r])}; 12^{([p, q, r])})\) with symmetry group \#632. Now, if we consider \( H_2 = (PQ, PR) \cong 632 \), \( [G_1 : H_2] = 2 \) we obtain \( \text{Stab}_{H_2}(t) \cong [e] \) and \( \text{Stab}_{G_1}(t) \cong D_1 \) again, taking \( x \) as the incenter we obtain tiling No. 52 \((3^{([p, q, r])}; 12^{([p, q, r])}; 12^{([p, q, r])})\). Now, taking \( x \) lying on an axis of reflection symmetry belonging to \( \text{Stab}_{G_1}(t) \) again yields a tiling belonging to family No. 53 \((3^{([p, q, r])}; 12^{([p, q, r])}; 12^{([p, q, r])})\). If we take \( x \) not lying on the axis of reflection symmetry we obtain tilings belonging to family No. 55 \((3^{([p, q, r])}; 12^{([p, q, r])}; 12^{([p, q, r])})\) with symmetry group 632. Observe that two subgroups \( H_1 \) and \( H_2 \) of \( G_1 \) may result in isocoronal tilings belonging to the same family of isocoronal tilings. So, in total, from one isohedral tiling \( T_1 \), we get four families of isocoronal tilings with the type of isocoronal tiling having a vertex corona consisting of a 3-gon and two 12-gons of varying symmetry-group types.

In the classification scheme of monocoronal isogonal tilings by Frettlo¨h & Garber (2015), included are tilings Nos. 48, 54 and 55. The others they do not list but instead consider these as limiting cases. Take for instance the vertex corona consisting of a 4-gon, a 6-gon and a 12-gon, where Frettlo¨h and Garber list only tiling No. 48 \((4^{([p, q, r])}; 6^{([p, q, r])}; 12^{([p, q, r])})\). The limiting cases are, if \( x \) is the incenter of tile \( t \), then tiling No. 47 is obtained. Moreover, if \( x \) lies on any angle bisector of \( t \) then the tilings Nos. 49, 50, 51 result. In this sense, if we consider only the possibilities where \( H \) is a subgroup of \( G \) that acts transitively on the tiles in \( T \) such that \( \text{Stab}_{H}(t) = [e] \), \( t \in T \), and a point \( x \in \text{Int}(t) \) is chosen to not lie on the axis of reflection symmetry in \( \text{Stab}_{G}(t) \), the center of rotational symmetry in \( \text{Stab}_{G}(t) \), the angle bisector of \( t \) or the incenter of \( t \), then we will arrive at a list of isocoronal tilings excluding the limiting cases.

**Theorem 4.2.** In \( E^2 \), there exist exactly eleven (11) edge-to-edge isocoronal tilings by regular polygons which can be obtained from the eleven (11) edge-to-edge tile-transitive

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**Figure 8 (continued)**

Isocoronal tilings of \( E^2 \).
tilings by tangential polygons with regular vertices, and conversely (Fig. 9).

The edge-to-edge isocoronal tilings by regular polygons in $\mathbb{E}^2$ are well known. These tilings are usually called the Archimedean tilings or uniform tilings (Grünbaum & Shephard, 1987), and include the regular tilings. Isohedral tilings by tangential polygons with regular vertices in $\mathbb{E}^2$ are often called the Laves tilings (Grünbaum & Shephard, 1987).

Fig. 9 shows pairs of tilings in $\mathbb{E}^2$ determined by Theorem 4.2. A pair consists of an isocoronal tiling by regular polygons (blue) and the corresponding isohedral tiling by tangential polygons with regular vertices (black). An isocoronal tiling is obtained from the tile-transitive tiling by connecting an edge the inccenters of the latter, and vice versa (also referred to as dual tilings). We indicate, for each pair, the number of sides of a regular polygon appearing in the vertex corona of an isocoronal tiling or, equivalently, the valences of a tile in a tile-transitive tiling, followed by the symmetry group of the tilings in Conway notation.

5. Tile-transitive isocoronal tilings

The result below outlines a method to arrive at isocoronal tilings that are tile-transitive. Among the Euclidean isocoronal tilings given in Fig. 8, there are 20 tilings that are tile-transitive (tilings that have labels marked blue). These tilings can be obtained using this result.

**Theorem 5.1.** Let $T$ be a tile-transitive $[p^n]$ tiling of $\mathbb{X}$ with symmetry group $G$. Suppose $H$ acts transitively on the tiles in $T$ and there exist (not necessarily distinct elements) $h_1, h_2, \ldots, h_n \in H$ (with axes of reflections or glide reflections passing through or with centers of rotations lying on a tile $t$ in $T$) such that $h_i h_{i-1} \cdots h_1 e$ where $h_i h_{i-1} \neq e, i = 2, \ldots, n$. Then there exists an edge-to-edge tile-transitive isocoronal tiling by $p$-gons with symmetry group $G$.

**Proof.** Consider $x \in \text{Int}(t)$ and form $Hx$. Following the arguments in Theorem 3.1 for $k = 1$, we obtain an isocoronal tiling $T'$ with the vertex corona of a given vertex consisting of $n$ $p$-gons.

Consider $h_1, h_2, \ldots, h_n \in H$ (with axes passing through or with centers lying on $t$) such that $h_i h_{i-1} \cdots h_1 e$ where $h_i h_{i-1} \neq e$. Let $P$ be a $p$-gon in the vertex corona of $x$. We have $h_i P$ a $p$-gon adjacent to $P$; $(h_i h_1 P)$ adjacent to $h_1 P$; and so on until $(h_n h_{n-1} \cdots h_1 P) = P$. So there will be $n$ congruent $p$-gons in the vertex corona of $x$. Since $T'$ is isocoronal then $T'$ consists of congruent $p$-gons.

Let $P_1$ and $P_2$ be any two $p$-gons in $T'$. Consider vertex coronae $A$ and $B$ in $T'$ containing $P_1$ and $P_2$, respectively. By the construction of $T'$, there exists $h \in H$ such that $B = h A$. It follows that there exists a $p$-gon $P'$ in $B$ such that $P' = h P_1$. Note that $P'$ and $P_2$ are among the congruent $p$-gons in $B$. So there exists $h' \in H$ such that $P_2 = h' P'$. Thus $P_2 = (h' h) P_1$. Hence $T'$ is tile-transitive. □

Consider the Euclidean tile-transitive isocoronal tiling ($6^3$) shown in Fig. 10(c) obtained from the $[6^3]$ tiling [Fig. 10(a)], which has symmetry group $G = \langle P, Q, R \rangle \cong \ast 632$. We consider the subgroup $H = \langle PR, QPR, RQPQP \rangle \cong \ast 2p_2$ of $G$ where $[G : H] = 6$. Take the elements $h_1, h_2, h_3$ of $H$ such that $(h_i h_j) \neq t$, $t$ is a tile in the $[6^3]$ tiling. In Fig. 10(b), $h_1 = PR$, $h_2 = QPRQ$ and $h_3 = RQPQR$ are twofold rotations whose centers (yellow) are shown. In the vertex corona of $x \in \text{Int}(t)$, take the 6-gon $P$ [Fig. 10(c)]. $h_1 P$ is a 6-gon adjacent to $P$; $(h_1 h_2) P$ is adjacent to $h_1 P$, and $(h_1 h_2 h_3) P = P$. We obtain three congruent 6-gons in the vertex corona of $x$.

As a second example, consider the hyperbolic tile-transitive isocoronal ($8^3$) tiling in Fig. 11(c) obtained from the $[8^3]$ tiling by regular 4-gons shown in Fig. 11(a) with symmetry group $G = \langle P, Q, R \rangle \cong \ast 842$. Consider $H = \langle RP, ORQP, RQOROP \rangle \cong 222 \times$, a subgroup of index 8 in $G$. Its generators $h_1 = RP$, $h_2 = QRPQ$ are twofold rotations with centers (yellow) shown in Fig. 11(b). The generator $h_1 P$ is $PQRQPQP$ of $H$ is a glide reflection with axis shown in Fig. 11(b). In the vertex corona of $x \in \text{Int}(t)$, take the 8-gon $P$ [Fig. 11(c)]. $h_1 P$ is an 8-gon adjacent to $P$; $(h_1 h_2) P$ is adjacent to $h_1 P$; and $(h_1 h_2 h_3) P = P$ where $h_2 = h_3$. We obtain four congruent 8-gons in the vertex corona of $x$. This example shows that an 8-gon can tile $H^2$ to obtain a tile-transitive isocoronal tiling. The $8^3$ tiling has symmetry group $G^* = H \cong 222 \times$.

6. Conclusion and future directions

In this article, we presented an approach to systematically arrive at planar edge-to-edge $k$-isocoronal tilings from tilings satisfying tile-transitivity properties.
under subgroups of their symmetry groups. The method not only facilitates the classification of tilings with \( k \) orbits of vertex coronae under the symmetry groups, but also provides a way to determine tilings with vertex-\( k \)-transitivity properties, such as the \( k \)-uniform tilings. It also describes the symmetry groups of the tilings that arise. The method is encapsulated in Theorem 3.1 and Theorem 3.2, the latter focusing on \( k \)-isocoronal tilings by regular polygons. To illustrate the method, we give some Euclidean and hyperbolic examples of \( k \)-isocoronal and \( k \)-uniform tilings that are not provided in the literature.

To arrive at a \( k \)-isocoronal tiling, the process involves a starting tiling \( T \) satisfying the property that there exists a subgroup \( H \) of its symmetry group \( G \) that forms \( n \) orbits of tiles, \( n \geq k \). The suitable subgroups of \( G \) to use are determined using \( GAP \) and tested to satisfy the orbit conditions using dynamic geometry software. The possibilities for a starting tiling in \( \mathbb{E}^2 \) would be one of the tile-transitive tilings provided

Figure 10
Construction of a tile-transitive isocoronal tiling (6\(^4\)) from the [6\(^3\)] tiling.

Figure 11
Construction of a tile-transitive isocoronal tiling (8\(^4\)) from the [8\(^4\)] tiling. Tilings are exhibited in the Poincaré disc model of \( \mathbb{H}^2 \).
in Schattschneider & Dolbilin (1998). In $\mathbb{H}^2$, one has to rely on known families of tile-transitive tilings such as those provided in Mitchell (1995), Conway et al. (2008). It is also possible to start with a tile-$s$-transitive tiling. An important reference in arriving at tile-$s$-transitive tilings is Huson (1993) where the construction of a tile-$s$-transitive tiling of $\mathbb{E}^2$ or $\mathbb{H}^2$ from a tile-$(s - 1)$-transitive tiling using the concept of gluing and splitting of tiles is discussed. It is as of this writing the only method that could provide a complete list of tile-$s$-transitive tilings.

The isocoronal tilings in $\mathbb{E}^2$ that we derived in Section 4 are a special case of applying the method for $k = 1$. We also presented a theorem in Section 5 that deals with the construction of tile-transitive isocoronal tilings. This result helps determine what single tile shapes give rise to tile- and vertex-transitive tilings of $\mathbb{E}^2$ and $\mathbb{H}^2$. As shown in Fig. 8, there are 20 tile-transitive isocoronal tilings of $\mathbb{E}^2$. Classes of hyperbolic tilings can be derived using the theorem that adds to the classification by symmetry groups of hyperbolic tile-transitive tilings given in Conway et al. (2008). The symmetry group of the tiling given in Fig. 11(c) is an example not in the list.

The method we presented here in arriving at $k$-isocoronal tilings can also be used to generate 2-periodic tilings with transitivity $k l m$, tilings that are important and are targets for design synthesis. Tilings with transitivity $k l m$ are vertex-$k$-transitive, edge-$l$-transitive and tile-$m$-transitive. Tilings with transitivity $k - 1 m$ are rare (Delgado-Friedrichs & O’Keeffe, 2017b). Following our approach, we discovered the $(4^3; 4^4; 4^4)$ tiling shown in Fig. 1 and discussed in Section 3, a tiling with transitivity 322 not included in the list of tilings with transitivity 32$m$ in Delgado-Friedrichs & O’Keeffe (2017b). Another example that we were able to derive is the $(4^4; 4^4; 4^4)$ tiling shown in Fig. 12(c) with transitivity 432 and symmetry group $G^* = \langle P, Q, R, QPRPQRPRQ \rangle \cong p4mm$. This has been obtained from the [44$^4$] tiling with symmetry group $G = \langle P, Q, R \rangle \cong p4mm$ [Fig. 12(a)] where the index-8 subgroup $G^*$ of $G$ is used.

Generating a $k$-isocoronal tiling $T^*$ with additional transitivity properties (e.g. tilings with transitivity $k k - 1 m$) using our given method would have to entail looking closely at the specific type of subgroup $H$ of the symmetry group of the starting tiling that will form $k$ orbits of tiles and at the same time determining carefully the choice of the interior points of the representative tiles in each $k$ orbit so that only $k - 1$ kinds of edges will be formed in $T^*$. This is not an easy task and it will help for future work to formulate a computer program to facilitate more systematically the construction of these types of tilings and of $k$-isocoronal tilings in general.

There are several other directions in which this study can be continued. One is to study planar tilings with congruent edge coronae and determine if these tilings are edge-transitive. Frettlöh & Garber (2015) discussed a method of obtaining monocoronal tilings in $\mathbb{E}^2$ that are not vertex-transitive. It will also be interesting to develop a method to characterize these types of tilings in $\mathbb{E}^2$. In general, monocoronal tilings that are non-periodic or aperiodic are not vertex-transitive. It will also be worthwhile studying non-periodic monocoronal tilings other than the tilings given in Frettlöh & Garber (2015), Goodman-Strauss (2009). One may adapt the method given in this work in arriving at isocoronal tilings of $\mathbb{E}^2$ or $\mathbb{H}^3$. In general, one can pursue the characterizations of periodic and non-periodic monocoronal tilings of $n$-dimensional Euclidean and hyperbolic space.

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