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Local and global color symmetries of a symmetrical pattern

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This study addresses the problem of arriving at transitive perfect colorings of a symmetrical pattern $P$ consisting of disjoint congruent symmetric motifs. The pattern $P$ has local symmetries that are not necessarily contained in its global symmetry group $G$. The usual approach in color symmetry theory is to arrive at perfect colorings of $P$ ignoring local symmetries and considering only elements of $G$. A framework is presented to systematically arrive at what Roth [Geom. Dedicata (1984), 17, 99–108] defined as a coordinated coloring of $P$, a coloring that is perfect and transitive under $G$, satisfying the condition that the coloring of a given motif is also perfect and transitive under its symmetry group. Moreover, in the coloring of $P$, the symmetry of $P$ that is both a global and local symmetry, effects the same permutation of the colors used to color $P$ and the corresponding motif, respectively.

1. Introduction

The theory of color symmetry was first introduced in the 1950s by Shubnikov and has spawned research in different areas of symmetry theory (Senechal, 1988). The study of color symmetry uses several branches of mathematics, such as algebra, discrete geometry and number theory. It has diverse applications in the physical sciences, art and culture.

The notion of color groups, for example, is manifested in the study of crystal structures (De Las Peñas & Basilio, 2004; De Las Peñas & Felix, 2007; Laigo et al., 2009; Shubnikov & Koptsik, 1974) and phase transitions (Kotsev & Rustamov, 1983). Lifshitz (1997) discussed the applications of color symmetry to periodic and quasiperiodic crystals, including magnetic point groups of quasicrystals. Bugarin et al. (2008, 2013), De Las Peñas & Bugarin (2011), De Las Peñas et al. (2011) discussed Bravais colorings of planar modules associated with quasicrystallographic and non-periodic structures. There are also several applications on projections for crystal structure determination (Koptsik, 1975), twinning (Holser, 1961), magnetic structures (Kopsky, 2015; Litvin, 2014) and ferroelectric structures (Shuvalov & Belov, 1962). Color symmetry was used to come up with geometric models of nanotubes (De Las Peñas, Loyola et al., 2014; Loyola et al., 2012) and nanotori (Loyola et al., 2015), and was used as a method to arrive at crystallographic origami (De Las Peñas et al., 2015; Taganap et al., 2014). In art and culture, studies pertaining to color repetition have been carried out in the study of San Ildefonso pottery designs (Crowe & Washburn, 1985), sandals of the Basketmaker and Pueblo peoples (Campbell, 1989; Teague & Washburn, 2013), Moorish ornaments (Grünbaum et al., 1986) and Peruvian textiles.

One of the problems in color symmetry is the classification and enumeration of colored symmetrical patterns. Several papers in the literature are dedicated to addressing this problem (see De Las Peñas et al., 1999a,b, 2006; Evidente, 2012; Felix & Junio, 2015; Loeb, 1978; Schwarzenberger, 1984; Senechal, 1979 and references therein). In these studies, colors are added to different elements of the pattern (e.g. tiles, edges or vertices of a tiling) ignoring possible local symmetries, and considering only the action of the pattern’s symmetry group (or its subgroup) on the set of colors.

Roth (1984) presented a theory of local color symmetry and introduced the idea of a ‘coordinated coloring’ of a symmetrical pattern, where the coloring possesses at least one non-identity symmetry that effects the same permutation of the colors both as a global and local color symmetry. He discussed obtaining a coordinated coloring of a symmetrical pattern consisting of disjoint congruent motifs, where the pattern may possess local symmetries that are not global symmetries. He focused on a few examples of coordinated colorings of patterns where a region that is assigned a color has as stabilizer the identity in the symmetry group of the pattern or motif. A method to arrive at these colorings was not provided.

This study contributes to the growing body of literature in color symmetry theory by continuing the work started by Roth (1984) on local color symmetry theory. We present a systematic method to arrive at coordinated colorings of a more general class of symmetrical patterns, to include patterns wherein the region that is assigned a color has non-identity symmetries. To arrive at this method, an analysis of the properties of coordinated colorings is carried out in detail, including a characterization of the relations between global and local color symmetries of symmetrical patterns.

In addition to its numerous applications to other disciplines – chemistry, physics, culture – color symmetry continues to play a role in the larger area of symmetry theory. In crystallography one of the possible applications is in crystal engineering – in the engineering of a building block towards the design of crystalline solids with predefined and desired aggregation of molecules and ions adhering to local and global symmetry properties. Another is in the fragmentation of crystals into domain structures, satisfying local and global symmetry properties. What remains as a challenging problem in symmetry theory is the understanding of the relation of global and local order, and to date it has transcended notions of periodic and crystal structures, to monoperiodic, non-periodic and highly ordered structures. Color symmetry theory and the techniques that have been developed through the years are intended to help clarify this relation. Here we address the relation of global and local color symmetry, hoping it can clarify particular questions on levels of symmetry and order, with regards to periodic structures.

The organization of this paper is as follows. In Section 2, we give preliminary notions on color symmetry used in the paper. In Section 3 we present the class of symmetrical patterns considered in this work and discuss global and local symmetries of symmetrical patterns. Section 4 defines coordinated colorings and gives the setting for which coordinated colorings of symmetrical patterns are considered. Section 5 discusses the properties of coordinated colorings and provides the method for obtaining coordinated colorings. Section 6 gives applications of the method to arrive at coordinated colorings of planar patterns. Section 7 examines the chromatic/achromatic properties of the symmetries and partial operations of a coordinated coloring, and Section 8 gives a summary and outlines the future direction of the work.

2. Preliminaries

Let \( X \subset \mathbb{E}^2 \) (Euclidean plane) be the set consisting of objects to be colored. If \( S = \{C_1, C_2, \ldots, C_n\} \) is a set of \( n \) colors, then an \( n \)-coloring of \( X \) is a surjective (or onto) map \( f : X \to S \). Each object \( x \in X \) is assigned a color \( f(x) \) in \( S \). The \( n \)-coloring of \( X \) is given by the partition \( P = \{P_1, P_2, \ldots, P_n\} \) of \( X \) where two elements of \( X \) belong to the same set in \( P \) if and only if they are assigned the same color. Each set \( P_i \) in the partition of \( P \) is represented by a color.

Let \( G \) be the symmetry group of \( X \), where \( G \subset \text{Isom}(\mathbb{E}^2) \), the group of isometries in \( \mathbb{E}^2 \). An element of \( G \) that effects a permutation of the colors in the \( n \)-coloring of \( X \) is called a color symmetry. If all the elements of \( G \) are color symmetries, then the coloring of \( X \) is called a perfect coloring.

The action of \( G \) on the set \( S \) of colors used in a perfect \( n \)-coloring of \( X \) induces a homomorphism \( \Pi : G \to A(S) \) where \( A(S) \) is the group of permutations of colors in \( S \). If \( g \in G \), \( \Pi(g) \) is the corresponding permutation of colors in \( S \) induced by \( g \), called the color permutation corresponding to \( g \). The homomorphism \( \Pi \) that results from the action of \( G \) on \( S \) is also referred to as a permutation representation \( \Pi \) of \( G \) in \( S \) (Roth, 1984).

A perfect coloring of \( X \) is transitive if the symmetry group \( G \) of \( X \) acts transitively on the set \( S = \{C_1, C_2, \ldots, C_n\} \) of colors used in the coloring. That is, if \( C_a \) and \( C_b \), \( a, b \in \{1, 2, \ldots, n\} \), are any two colors in \( S \), then there is an element in \( G \) that sends \( C_a \) to \( C_b \). Given a transitive perfect coloring of \( X \) with colors coming from \( S \), the homomorphism \( \Pi \) that results from the action of \( G \) on \( S \) is called a transitive permutation representation.

Consider a symmetrical pattern \( \mathcal{P} \), which is a regular hexagon divided into 12 regions, each of which has been assigned a color from the set \( S = \{b – \text{blue}, r – \text{red}, y – \text{yellow}\} \). To obtain the 3-coloring of \( \mathcal{P} \) shown in Fig. 1(a). When uncolored, the symmetry group \( G \) of \( \mathcal{P} \) is the dihedral group of order 12; \( G \cong S_3 \) generated by the 60° counterclockwise rotation \( e^6 \in G \) about the center of \( \mathcal{P} \) with coordinates 0,0 (denoted by \( e^6 \)) and reflection \( m_{100} \) with axis the horizontal line passing through 0,0. Observe that the 3-coloring of
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\[ \mathcal{P} \text{ given in Fig. 1(a) is perfect. The rotational symmetry } 6^+ \text{ sends blue to red, red to yellow, and yellow to blue. On the other hand, the reflection } m_{[100]} \text{ interchanges blue and red, and fixes yellow. The action of } G \text{ on the set } S \text{ of colors induces a permutation representation } \mu \text{ where } \mu(6^+ 0,0) = (b \ r y) \text{ and } \mu(m_{[100]}) = (b \ r) \text{. This 3-coloring of } \mathcal{P} \text{ is a transitive perfect coloring. It can be checked that, given any two colors, there is a permutation representation of } G \text{ that sends the color pink to any of the colors blue, red or yellow.}

\]

It can be observed that the regions that make up \( \mathcal{P} \) in the above example consist of two orbits under \( G \). The kites make up one \( G \)-orbit \( X_1 \), and the trapezoids make up a second \( G \)-orbit \( X_2 \). In the transitive perfect coloring given in Fig. 1(a), kites and trapezoids belonging to \( X_1 \) and \( X_2 \), respectively, are assigned the same set of colors whereas in the non-transitive perfect coloring presented in Fig. 1(b), the kites and trapezoids do not share a color.

In general, to obtain a transitive perfect \( n \)-coloring of a symmetrical pattern \( \mathcal{P} \) (or a set \( X \)), consisting of objects that are colored) where there is more than one \( G \)-orbit of regions in \( \mathcal{P} \), only one set of colors, say \( S = \{C_1, C_2, \ldots, C_n\} \), is used to color perfectly each and every \( G \)-orbit of regions (Gentuyan & Felix, 2013). Suppose there are \( p \) \( G \)-orbits of regions in \( \mathcal{P} \) given by \( X_i, i = 1, 2, \ldots, p \). To color perfectly a \( G \)-orbit \( X_i \), we use a subgroup \( J \) of \( G \) if it contains \( \text{Stab}_g(x_i) = \{g \in G | gx_i = x_i\} \) where \( x_i \in X_i \). If color \( C_i \) is used to color \( x_i \), then \( C_i \) is also used to color all the regions belonging to \( Jx_i \), the \( J \)-orbit of \( x_i \). Then we use the same \( J \) in coloring perfectly another \( G \)-orbit \( X_j \) of regions, \( j \in \{1, 2, \ldots, p\}, j \neq i \), provided that \( J \) contains the stabilizer of a region \( x_j \) and \( x_i \), and we will also use the same color \( C_i \). Thus, color \( C_i \) is assigned to the elements in \( Jx_i \). We have \( C_i \) assigned to the regions in \( J[x_i, x_j] = Jx_i \cup Jx_j \). Continuing in the same manner in using \( J \) to color all \( G \)-orbits, we have \( J[x_1, x_2, \ldots, x_p] = Jx_1 \cup Jx_2 \cup \ldots \cup Jx_p \) which is also assigned color \( C_i \) and where \( \text{Stab}_g(x_i) \leq J, x_i \in X_i, i = 1, 2, \ldots, p \). The \( n \)-coloring of \( \mathcal{P} \) or \( X \) is described by the partition \( \mathcal{P} = \{g(x) | g \in G\} = \{g_1JT, g_2JT, \ldots, g_nJT\} = g_1JT \cup g_2JT \cup \ldots \cup g_nJT \) where \( \{g_1, g_2, \ldots, g_n\} \) is a complete set of left coset representatives of \( J \) in \( G \), \( T = \{x_1, x_2, \ldots, x_p\} \) and each \( gJT \) in \( \mathcal{P} \) is assigned the color \( C_i, l = 1, 2, \ldots, n \).

Referring to the transitive perfect coloring given in Fig. 1(a), if we consider \( x_i \in X_1 \) and \( x_j \in X_2 \), the coloring is obtained as follows. We have \( \text{Stab}_g(x_i) = \{1\} \) and \( \text{Stab}_g(x_j) = \{m_{[110]}\} \). Here \( 1 \) is the identity element, and \( m_{[110]} \) and \( m_{[100]} \) refer to the reflections with axes the diagonal lines passing through 0,0 as shown. A subgroup \( J \) of \( G \) that contains \( \{1\} \) and \( \{m_{[110]}\} \) is \( J = \{1, m_{[100]}, m_{[110]}\} = \{2m\} \). The left coset decomposition of \( G \) with respect to \( J \) determines the elements of \( \mathcal{P} \) assigned with the same color. A complete set of left coset representatives of \( J \) in \( G \) is \( \{1; 3^+ 0,0; 3^+ 0\} \). The elements \( 3^+ \) and \( 3^- \) are 120° counterclockwise and clockwise rotations about the coordinates 0,0, respectively. We assign the respective colors blue, red and yellow to \( JT, JT \) and \( JT \) where \( T = \{x_i, x_j\} \). The transitive \( \beta \)-coloring of \( \mathcal{P} \) is described by the partition \( JT \cup JT \cup JT \).

3. Local and global color symmetries of a symmetrical pattern

We first describe the class of symmetrical patterns under consideration in this study. Consider a symmetrical pattern \( \mathcal{P} \) consisting of disjoint congruent copies of a symmetrical pattern \( M \). We will refer to \( M \) and its congruent copies as motifs of \( \mathcal{P} \). For two motifs \( M_1, M_2 \) to be congruent, we mean there is \( g \in \text{Isom}(\mathbb{E}^2) \) such that \( M_2 = gM_1 \). The symmetry group \( G \) of \( \mathcal{P} \) will be referred to as the global symmetry group of \( \mathcal{P} \) and its elements are called global symmetries of \( \mathcal{P} \). We assume \( G \) acts transitively on the motifs so we may write \( \mathcal{P} = \bigcup_{g \in G} gM \). Note that we assume \( M = eM \), where \( e \) is the identity in \( G \). The pattern \( \mathcal{P} \in \mathbb{E}^2 \) here is assumed to be periodic, that is there is a nonzero vector \( z \in \mathbb{E}^2 \) such that \( \mathcal{P} + z = \mathcal{P} \).

As will be verified in the proposition below, if \( K \) is the symmetry group of the motif \( M \) of \( \mathcal{P} \), then the symmetry group of another motif \( gM \in \mathcal{P} \) is \( gKg^{-1} \). The group \( gKg^{-1} = \{gkg^{-1} | k \in K\} \) is called the conjugate of \( K \) by \( g \). We call the local symmetry group of \( \mathcal{P} \) and its elements are called local symmetries of \( \mathcal{P} \).

Note that \( G, gKg^{-1} \subset \text{Isom}(\mathbb{E}^2) \), so the operation in \( G \) or \( gKg^{-1} \) is a composition of isometries. If \( \alpha_1, \alpha_2 \in G \) or \( gKg^{-1} \), then \( \alpha_1 \circ \alpha_2 \) is an isometry. If \( \alpha_1, \alpha_2 \in G \) or \( gKg^{-1} \), then \( \alpha_1 \circ \alpha_2 \) is an isometry.

**Proposition 3.1.** If \( K \) is the symmetry group of the motif \( M \) of \( \mathcal{P} \), then the symmetry group of another motif \( gM \) is \( gKg^{-1} \).

**Proof.** Suppose \( K \) is the symmetry group of the motif \( M \). Now for a motif \( gM \) of \( \mathcal{P} \) and \( \varphi \in \text{Isom}(\mathbb{E}^2) \) (group of isometries of the Euclidean plane), \( \varphi(gM) = gM \) if and only if \( (g^{-1} \varphi g)M = M \) if and only if \( g^{-1} \varphi g \in K \) or \( \varphi \in gKg^{-1} \). \( \square \)
In the symmetrical pattern $P$ under consideration, a motif may contain symmetries that are not elements of $G$, as will be seen in the following example.

Fig. 2 presents a frieze pattern $P_1$ consisting of disjoint congruent copies of the square $M$. The squares are the motifs of $P_1$. The symmetry group of $M$ is $K = \{4^Z, 0,0; \{m_{102}, m_{103}\}\} \cong 4mm$ (dihedral group of order 8) where $4^Z$ is the 90-degree counterclockwise rotation at the center of $M$ with coordinates 0,0, and $m_{102}$ is the reflection with horizontal line passing through 0,0, and $m_{103}$ is the reflection with line passing through $z(1,0)$. The colors of $G$ are called the global symmetries of $G$. The elements of $G$ are among the local symmetries of $P_1$. That is, some of the local symmetries of $P_1$ are not global symmetries of $P_1$. In this paper, we will look closely at symmetrical patterns such as this, patterns that have local symmetries that are not global symmetries.

4. Coordinated colorings of a symmetrical pattern

This section describes the setting in obtaining coordinated colorings. Consider a symmetrical pattern $P$ with symmetry group $G$ consisting of disjoint congruent motifs $gM(g \in G)$. Suppose each motif $gM$ is divided into $r$ congruent regions. We assume that each $gM$ is partitioned such that its symmetry group $gKg^{-1}$ acts transitively on these regions. We may write $gM$ as $gM_1 \cup gM_2 \cup \ldots \cup gM_r$, and $P$ as $\bigcup_{g \in G} \bigcup_{j=1}^{r} gM_j$. We now construct an $n$-coloring of $P$ such that each region $gM_j$ is given a color from a set $S = \{C_1, C_2, \ldots, C_n\}$ of $n$ colors. The $n$-coloring of $P$ defines a surjective function $f : P \rightarrow S$ which assigns to each $gM_j$ a color $f(gM_j) \in S$.

In this $n$-coloring of $P$, we make several assumptions. First, we assume that the $n$-coloring of $P$ is perfect, that is, every element of $G$ permutes the colors of the $n$-coloring. We also assume that $G$ acts transitively on $S$, the set of colors in the $n$-coloring. Moreover, in a motif of $P$, given by $gM$, we assume we have a perfect coloring where every element of $gKg^{-1}$ permutes the colors in $S' \subseteq S$, the colors used in the coloring of $gM$. It follows that $gKg^{-1}$ acts transitively on $S'$ since $gKg^{-1}$ acts transitively on $\{gM_1, gM_2, \ldots, gM_r\}$. Consequently, there exists a transitive permutation representation $\mu$ of $G$ on $S$ and a transitive permutation representation $\tau_\mu$ of $gKg^{-1}$ on $S'$. In this sense, we are considering a perfect and transitive coloring of $P$ such that each motif $gM$ is also colored perfectly and transitively with respect to its symmetry group $gKg^{-1}$. The elements of $G$ are called global color symmetries of $P$ and the elements of $gKg^{-1}(g \in G)$ are called local color symmetries of $P$.

If $g^* \in G \cap gKg^{-1}$, then $g^*$ is both a global and a local color symmetry of $P$. As a global color symmetry, $\mu(g^*)$ is the color permutation corresponding to $g^*$ and as a local color symmetry, $\tau_\mu(g)$ is the color permutation corresponding to $g^*$.

We will assume that as a global symmetry and as a local symmetry, $g^*$ affects the same permutation of the colors in the coloring of $P$ and $gM$, respectively. This means that $\mu(g^*)|S'_g = \tau_\mu(g^*)$. Note that, in general, not all colors in $S$ are used to color the motif $gM$. So that in considering color permutations of $g^*$ as a global and local symmetry we restrict $\mu(g^*)$ to colors in $S'_g$. We write this as $\mu(g^*)|S'_g$. We now state formally the definition of a coordinated coloring of $P$ as follows.

**Definition 4.1.** Consider a symmetrical pattern given by $P = \bigcup_{g \in G} \bigcup_{j=1}^{r} gM_j$, with symmetry group $G$. Suppose each motif $gM(g \in G)$ has symmetry group $gKg^{-1}$. Consider an $n$-coloring of $P$ given by $f : P = \bigcup_{g \in G} \bigcup_{j=1}^{r} gM_j \rightarrow S$ where $S = \{C_1, C_2, \ldots, C_n\}$ is a set of $n$ colors and each $gM_j$ is assigned a color $f(gM_j) \in S$. Then the coloring is coordinated if there exists a transitive permutation representation $\mu$ of $G$ on $S$ and a transitive permutation representation $\tau_\mu$ of $gKg^{-1}$ on $S'_g$ (set of colors in $gM$) such that $\mu(g)|S'_g = \tau_\mu(g)$ for all $g^* \in G \cap gKg^{-1}$.

To illustrate a coordinated coloring of a symmetrical pattern, consider the frieze pattern $P_1$ given in Fig. 2 and suppose each square motif is divided into four congruent isosceles triangles. A 4-coloring of $P_1$ is shown in Fig. 3 where an isosceles triangle is assigned a color from the set $S = \{b – blue, y – yellow, r – red, g – green\}$. The triangles in each motif are given two colors: either blue and red, or yellow and green. Every element of $G = \{2^1, 0; m_{102}, m_{103} ; z(1,0)\} \cong 2mm$ permutes the colors in the 4-coloring of $P_1$. As a result of the action of $G$ on $S$ we obtain the permutation representation $\mu$ where $\mu(m_{102}) = (b)$, $\mu(m_{103}) = (y \ g)$ and $\mu(z) = (b \ r \ y \ g)$. It can be checked that given two colors in $S$, there exists an element in $G$ that sends the colors to each other. Thus, $G$ acts transitively on $S$. We also note that in this coloring, every element of $gKg^{-1}$ permutes the colors assigned to the triangles.
in the motif $gM$. Some examples include: every element of $K = \{4^* \, 0, 0; m_{10} \} \cong 4mm$ permutes the colors in $S_1 = \{\text{blue, red}\}$, the set of colors used in the coloring of the motif labeled $M$ giving rise to the permutation representation $\tau_1$ where $\tau_1(4^*) = (b, r)$ and $\tau_1(m_{10}) = (b)$; every element of $\kappa(zKz^{-1} = \{4^* \, 0, 0; m_{10} \} \cong 4mm$ permutes the colors in $S_1 = \{\text{green, yellow}\}$, the set of colors used in the coloring of the motif $zM$ resulting in the color permutations $\tau_1(4^*) = (y, g)$ and $\tau_1(m_{10}) = (b)$.

It can be checked that for $g^* \in G \cap gKg^{-1}$, we have $\mu(g^*)|S_1 = \tau_1(g^*)$. Note for example that $G \cap K = \{1_{id}; 20, 0; m_{10}; m_{01}\}$. We obtain the following: $\mu(20, 0)|S_1 = \tau_1(x, 20, 0) = (b)$, $\mu(m_{10})|S_1 = \tau_1(x, m_{10}) = (b)$ and $\mu(m_{01})|S_1 = \tau_1(x, m_{01}) = (b)$. Also, $G \cap \kappa(zKz^{-1} = \{1_{id}; 20, 0; m_{10}; m_{01}\}$. Note that $2, 10$ and $m_{01}$ are, respectively, the 180° rotation about the coordinates 1,0, the center of $zM$ and reflection with axis the vertical line passing through 1,0. We have $\mu(21, 0)|S_1 = \tau_1(x, 21, 0) = (y)$, $\mu(m_{10})|S_1 = \tau_1(x, m_{10}) = (b)$ and $\mu(m_{01})|S_1 = \tau_1(x, m_{01}) = (y)$. The 4-coloring shown in Fig. 3 is a coordinated 4-coloring.

5. Framework for obtaining coordinated colorings of a symmetrical pattern

In this section, a method will be presented for obtaining coordinated n-colorings of a symmetrical pattern $P = \bigcup_{g \in G} P_g$, where a region $gM$ in $P$ is given a color from a set $S$ of $n$ colors. We first discuss every component that comes into a coordinated coloring of $P$ and characterize this completely.

The following result from Evidente (2012) will be helpful in our discussion.

**Theorem 5.1.** (Evidente, 2012) Let $\mathbb{X}$ be a set and suppose $G^*$ is a group that acts transitively on $\mathbb{X}$.

(i) If $\{P_1, P_2, \ldots, P_n\}$ is a coloring of $\mathbb{X}$ for which elements of $G^*$ permute the colors, then for every $x \in \mathbb{X}$ there exists $H^* \leq G^*$ such that $\text{Stab}_{G^*}(x) \leq H^*$, and the coloring is described by the partition $\{g(x)|H^*|g^* \in G^*\}$.

(ii) Let $x \in \mathbb{X}$ and $H^* \leq G^*$ such that $\text{Stab}_{G^*}(x) \leq H^*$ and $|G^*: H^*| = n < \infty$. Then the partition $\{g(x)|H^*|g^* \in G^*\}$ is an n-coloring of $\mathbb{X}$ for which elements of $G^*$ permute the colors.

We now characterize the colorings of the motifs in a coordinated coloring of $P$. We have the following result.

**Theorem 5.2.** Consider a coordinated coloring of $P$ where a coloring of the motif $M$ has $n^*$ colors. Then for every region $m$ in $M$ there exists $L \leq K, [K : L] = n^*$ such that $\text{Stab}_K(m) \leq L$, and the coloring of $M$ is described by the partition $Q = \{kLm|k \in K\}$. Moreover, the coloring of $gM$ ($g \in G, g \neq e)$ is described by the partition $Q_g = \{g^*kLg^{-1}|m'|k' \in gKg^{-1}\}$, $\text{Stab}_{gKg^{-1}}(m') \leq gLg^{-1}$.

**Proof.** It can be recalled that in a coordinated coloring of $P$ the elements of the symmetry group $K$ of the motif $M$ act transitively on the regions in $M$ and the elements of $K$ permute the colors of the coloring of $M$. Suppose a coloring of $M$ has $n^*$ colors. Then from Theorem 5.1(i), for every region $m$ in $M$ there exists $L \leq K, [K : L] = n^*$ such that $\text{Stab}_K(m) \leq L$, and the coloring is described by the partition $Q = \{kLm|k \in K\}$. Now corresponding to $gM$ we have the partition $Q_g = \{g(kLm)|k \in K\} = \{g^*kLg^{-1}|m'|k' \in gKg^{-1}\}$ where $m' = gm \in gM, [gKg^{-1} : gLg^{-1}] = n^*$. If $\text{Stab}_K(m) \leq L$ then we obtain $g(\text{Stab}_K(m))g^{-1} \leq gLg^{-1}$. It follows that $\text{Stab}_{gKg^{-1}}(m') = g(\text{Stab}_K(m))g^{-1} \leq gLg^{-1}$.

The above result describes the coloring of the motif $M$ and the consequent coloring of the motif $gM$. The theorem tells us that if we have a perfect $n^*$-coloring of $M$ where the elements of $K$ permute the colors, then we also have a perfect $n^*$-coloring of $gM$ where the elements of $gKg^{-1}$ permute the colors.

Now, the coordinated coloring of $P$ satisfies the condition that the elements of the symmetry group $G$ of $P$ permute the colors of the coloring and $G$ acts transitively on the set $S$ of colors used to color $P$. Suppose $G$ forms p orbits of regions, $X_p, i = 1, 2, \ldots, p$ in $P$. To realize a transitive perfect coloring of $P$, each $G$-orbit of regions $X_i$ is colored perfectly using the set $S$ of $n$ colors. This means that the $G$-orbits of regions in $P$ will share the same colors. The following theorem characterizes the partition that describes the transitive perfect colorings of $P$.

**Theorem 5.3.** Suppose $G$ forms p orbits of regions $X_p, i = 1, 2, \ldots, \overline{p}$ in $P$.

(i) Consider an $n$-coloring of $P$ for which elements of $G$ permute the colors and $G$ is transitive on the colors; then for every region $x_n \in P(x_n \in X_n, x_n \in \overline{1}, 2, \ldots, \overline{p})$, there exists a $J \leq G, [G : J] = n$ and the coloring is described by the partition $P = \{gJ|1 = 1, 2, \ldots, \overline{n}\}$ where $T = \{x_1, x_2, \ldots, x_n\}$, $x_i \in X_i$ with $\text{Stab}_G(x_i) \leq J, i = 1, 2, \ldots, \overline{n}$ and ($g_1 = e, g_2, \ldots, g_n$) is a complete set of left coset representatives of $J$ in $G$.

(ii) Let $J \leq G$ such that $[G : J] = n < \infty$ satisfying the condition that a conjugate of $\text{Stab}_G(x_i)$ is contained in $J$, $x_i \in X_i, i = 1, 2, \ldots, \overline{n}$. Then the partition $P = \{gJ|g \in G\}$ where $T = \{x_1, x_2, \ldots, x_n\}$ is a set of $G$-orbit representatives from each $X_i$ such that $\text{Stab}_G(x_i) \leq J$ is an n-coloring of $P$ for which elements of $G$ permute the colors $G$ is transitive on the colors.

**Proof.** (i) Consider an $n$-coloring of $P$ for which elements of $G$ permute the colors and $G$ is transitive on the colors. Let the n-coloring of $P$ be given by the partition $\{P_1, P_2, \ldots, P_n\}$. Let $x_n \in P$. Assume $x_n \in X_n$, $w \in \overline{1}, 2, \ldots, \overline{r}$. Suppose $P_i$ is assigned the color of $x_n$. Let $J = \text{Stab}_G(P_i)$. If $g \in \text{Stab}_G(x_n)$ then $gx_n = x_n \in P_i$. But $g$ permutes the colors so $gP_i = P_i$. It follows that $g \in J = \text{Stab}_G(P_i)$. Hence, $\text{Stab}_G(x_n) \leq J$. Now $G$ is transitive on the colors, so the $G$-orbits of regions share colors. This implies that for each $i \in \overline{1}, 2, \ldots, \overline{r}$, $i \neq w$, we can also find $x_i \in X_i$ having the same color as $x_w$. We can show using a similar argument that $\text{Stab}_G(x_i) \leq J$. 

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Let $T = \{x_1, x_2, \ldots, x_n, \ldots, x_p\}$. Now, $Jx_1 \cup Jx_2 \cup \ldots \cup Jx_n \cup \ldots \cup Jx_p = JT \subseteq P_1$.

We now show $P_1 \subseteq T$. Consider $x \in P_1$. Note that $x \in X_i$, $i \in \{1, 2, \ldots, p\}$. Consider $x_i \in X_i \cap T$. Since $x$ and $x_i$ are elements of the same $G$-orbit of regions in $P$, which is $X_i$, then there exists $g \in G$ such that $x = gx_i$. Since $x$, $x_i \in P_1$, and $g$ permutes the colors, we have $gP_1 = P_1$. Then $g \in J$, so $x = gx_i \in Jx_i \cup Jx_2 \cup \ldots \cup Jx_n \cup \ldots \cup Jx_p = JT$. Thus $P_1 \subseteq T$. Consider another color $P_2$ and $x \in P_2$. Then $x' \in X_i$, $i \in \{1, 2, \ldots, p\}$. Consider $x_i \in X_i \cap T$. Now $x'$ and $x_i$ are elements of $X_i$. Then there exists $g_2 \in G$ such that $x' = g_2x_i$. Since $g_2$ permutes the colors, then $P_2 = g_2P_1$ or $P_2 = g_2JT$. In this case $g_2 \in G \setminus J$; otherwise $g_2P_1 = P_1$ and we have a contradiction. We can continue using the same argument to show there exists $g_3 \in G \setminus (J \cup g_2J)$ such that $P_3 = g_3JT$ and so on, until we have $g_n \in G \setminus (J \cup g_2J \cup \ldots \cup g_{n-1}J)$ such that $P_n = g_nJT$.

Conversely, suppose $g_2 \in G \setminus J$. Now $g_2$ permutes the colors so that $g_2JT = g_2P_1 = P_1$ for some $l \in \{2, 3, \ldots, n\}$. Assume $g_2P_1 = P_2$. Now if we take $g_3 \in G \setminus (J \cup g_2J)$, we have $g_3JT = g_3P_1 = P_1$ for some $l \in \{3, 4, \ldots, n\}$ and, in particular, $g_3P_1 = P_2$. We continue in this manner until we obtain $g_nJT = g_nP_1 = P_n$ where we have $g_n \in G \setminus (J \cup g_2J \cup \ldots \cup g_{n-1}J)$. We then have $\{P_1, P_2, \ldots, P_n\} = [g_2JT] = \{1, 2, \ldots, n\}$ where $g_1 = e, g_2, \ldots, g_n$ is a complete set of left coset representatives of $J$ in $G$.

(ii) Let $J \leq G$ such that $[G : J] = n < \infty$ satisfying the condition that a conjugate of $\text{Stab}_G(x_i)$ is contained in $J$, $x_i \in X_i$, $i = 1, 2, \ldots, p$, that is, $g(\text{Stab}_G(x_i))^{-1}Jg \leq J$, $g \in G$. Consider $x_i' = gx_i$. This implies that $\text{Stab}_G(x_i') \leq J$. Let $T = \{x_1', x_2', \ldots, x_n\}$. Form the partition $P = [g_1JT] = JT \cup g_2JT \cup \ldots \cup g_nJT$ where $g_1 = e, g_2, \ldots, g_n$ is a complete set of left coset representatives of $J$ in $G$. Consider $S = \{C_1, C_2, \ldots, C_n\}$, a set of $n$ colors. To each set $g_1JT \in P$, assign the color $C_i$, $i = 1, 2, \ldots, n$. Consequently, $P = [g_1JT] \in G$ is an $n$-coloring of $P$ for which elements of $G$ permute the colors and $G$ acts transitively on the colors. $\square$

In a coordinated coloring of $P$, a symmetry that is both a global and local color symmetry effects the permutation of colors of $P$ and the corresponding motif $gM$ (assuming the symmetry is an element of $gK^{-1}$, respectively. The next result describes the relationship between the subgroups used in the partitions describing the coloring of $P$ and the coloring of $gM$ in a given coordinated coloring of $P$.

**Theorem 5.4.** Consider the coordinated $n$-coloring of $P$ given by the partition $P = \{P_1, P_2, \ldots, P_n\} = [g_1JT] = \{1, 2, \ldots, n\}$ where $T = \{x_1, x_2, \ldots, x_n\}, x_k \in X_i \cap P_1 (X_i$ a $G$-orbit in $P), \text{Stab}_G(x_i) \leq J = \text{Stab}_G(P_1), i = 1, 2, \ldots, p$ and $[g_1 = e, g_2, \ldots, g_n]$ is a complete set of left coset representatives of the subgroup $J$ in $G$. Then $J \cap gK^{-1} = J \cap T$, where $gK^{-1}$ is the symmetry group of a motif $gM$ containing a region $m \in P_1$ and the coloring of $gM$ is described by the partition $Q_k = \{kJm, k' \in gK^{-1}\}, \text{Stab}_G(gM) \leq J$ and $\overline{T} = \text{Stab}_G(gM)$.

**Proof.** Consider a coordinated $n$-coloring of $P$. Assume the $n$-coloring of $P$ for which elements of $G$ permute the colors and $G$ acts transitively on the colors is described by the partition $P = \{P_1, P_2, \ldots, P_n\} = [g_1JT] = \{1, 2, \ldots, n\}$ where $J = \text{Stab}_G(P_1)$. Consider a motif $gM$ of $P$ containing a region $m \in P_1$. Since we have a coordinated coloring, then the coloring of $gM$ is such that elements of its symmetry group $gK^{-1}$ permute the colors. Then for $m \in gM$ there exists $\overline{T} \leq gK^{-1}$ such that $\text{Stab}_G(gM) \leq \overline{T}$ and the coloring of $gM$ is described by the partition $Q_k = \{kJm, k' \in gK^{-1}\}$. Now, $\overline{T} = \text{Stab}_G(gM)$. It follows that $J \cap gK^{-1} = \text{Stab}_G(gM)$ and $\overline{T} = \text{Stab}_G(gM)$.

**Theorem 5.5.** Suppose the assumptions of Theorem 5.4 hold. Then the number $c$ of $(G \cap T)$-orbits of regions in an $T$-orbit of regions in $M$ given by $\overline{TM}$, where $m \in P_1 \cap gM$ ($g \in G$) and $\overline{T} = \text{Stab}_G(gM)$, is equal to

$$\frac{|\overline{T}|}{|\overline{T}|/|\text{Stab}_G(gM)|}.$$
Theorem 5.7. Suppose the assumptions of Theorem 5.4 hold. Then the number \( c \) of \((G \cap L)\)-orbits of regions in \( \bar{Lm} \) where \( m \in \{P_1 \cup gM \ (g \in G) \) and \( \bar{L} = \text{Stab}_{g_K}(P_1) \) divides \( p \), the number of \( G \)-orbits of regions in \( \mathcal{P} \).

Proof. Consider a coordinated \( n \)-coloring of \( \mathcal{P} \) described by the partition \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} = \{gJT\} | l = 1, 2, \ldots, n \) where \( J = \text{Stab}_G(P_1) \). Suppose the number of \( G \)-orbits of regions in \( \mathcal{P} \) is \( p \). We prove the result by constructing the set \( T \) consisting of a complete set of \( G \)-orbit representatives that will yield the given coordinated coloring. (Note that \( T \) consists of \( J \)-orbit representatives.) We first consider a motif \( M \) of \( \mathcal{P} \) with symmetry group \( K \) containing a region \( m_1 \in P_1 \). Consider \( L_1 = \text{Stab}_M(P_1) \). Construct \( T_1 \), a complete set of \((G \cap L_1)\)-orbit representatives in a \( L_1 \)-orbit of regions in \( M \), where the perfect coloring of \( M \) is described by the partition \( \mathcal{Q}_1 = \{kLm_1 | k \in K \} \). \( \text{Stab}_M(m_1) \leq L_1 \). Suppose \( |T_1| = c \). If \( c = |T_1| \), then we take \( T = T_1 \) and the conclusion follows.

Suppose \( c = |T_1| < p \). To complete the set of \( G \)-orbit representatives that will make up a possible composition of \( T \), we consider another motif \( g_2M \) of \( \mathcal{P} \), \( g_2 \neq e \), \( g_2 \notin J \) containing a region \( m_2 \in P_1 \). Consider \( L_2 = \text{Stab}_M(P_1) \). Now construct \( T_2 \), a complete set of \((G \cap L_2)\)-orbit representatives in a \( L_2 \)-orbit of regions in \( g_2M \), where the perfect coloring of \( g_2M \) is described by the partition \( \mathcal{Q}_2 = \{kLm_2 | k' \in g_Kg_2^{-1} \} \). \( \text{Stab}_M(m_2) \leq L_2 \). Since \( L_1 = g_2L'm_2' \), \( g' \in G \) thus, we have \( |T_2| = |T_1| = c \). If \( 2c = p \), then \( T = T_1 \cup T_2 \) and the conclusion is true.

If \( 2c < p \), we continue the process. We keep taking a \( g_qM \) of \( \mathcal{P} \), \( g_q \neq e \), \( g_q \notin J \) containing a region \( m_q \in P_1 \), \( q = 3, 4, \ldots, d \) to form \( T_1 \cup T_2 \cup \ldots \cup T_d \) where \( T_q \) is the complete set consisting of \((G \cap L_q)\)-orbit representatives in a \( L_q \)-orbit of regions in \( g_qM \), \( g_q \notin J \), \( |T_q| = c \). The perfect coloring of \( g_qM \) is described by the partition \( \mathcal{Q}_q = \{kLm_q | k' \in g_Kg_2^{-1} \} \). \( \text{Stab}_M(m_q) \leq L_q \) and \( L_q = \text{Stab}_M(P_1) \). Again, if \( cd = p \) then \( T = T_1 \cup T_2 \cup \ldots \cup T_d \) and the conclusion is true.

Now suppose \( T = T_1 \cup T_2 \cup \ldots \cup T_{p-1} \cup T \) where \( |T| = r \), \( r < c, T_d \supset \). That is, the number \( p \) of \( G \)-orbit representatives is such that \( p = cd - 1 + r \). Let \( x \in T \) \( \mathcal{P} \). Now \( x \in L \) \( m \) so that \( x \in P_1 \). Then \( x \in JT \) so we can write \( x = fT' \), \( T' \in T_1 \cup T_2 \cup \ldots \cup T_{d-1} \). Without loss of generality, suppose \( x \in T_1 \subset M \). Then \( f' \) sends a region \( T' \) of color \( P_1 \) in \( M \) to a region \( x \) of color \( P_1 \) in \( g_M \). Since the element \( f' \) permutes the colors, \( f' \) sends all the regions of color \( P_1 \) in \( M \) to regions of color \( P_1 \) in \( g_M \). The regions of color \( P_1 \) in \( M \) are the set \( (G \cap L) T_1 \), so we have \( (G \cap L) T_1 \subset g_M \). So that \( (G \cap L) T_1 \cup (G \cap L) T \) are regions of color \( P_1 \) in \( g_M \). Clearly \( |(G \cap L) T_1 \cup (G \cap L) T| > |(G \cap L) T_1| \). This is a contradiction since all the regions colored \( P_1 \) in each motif are congruent. Thus \( p = xK \) for some positive integer \( d \).

To clarify further the above result, we discuss the following example where \( c \) does not divide \( p \); in particular we have \( p = 3 \) and \( c = 2 \). Consider the frieze pattern \( \mathcal{P}_2 \) given in Fig. 4(a) consisting of disjoint congruent equilateral triangles where each equilateral triangle (motif) is divided into six congruent right triangles. The symmetry group of \( \mathcal{P}_2 \) is \( G = \{m_{010}; z(1, 0) \} \text{plm1} \) and the symmetry group of the motif labeled \( M \) is \( K = (3+0, 0); m_{010} \text{dihedral group of order 6} \) where \( 3^+ \) is the 120° counterclockwise rotation about the coordinates 0,0, the center of \( M, m_{010} \) is the reflection with axis the vertical line passing through 0,0 and \( z(1, 0) \) is the translation with vector \((1, 0)\). There are three \( G \)-orbits of regions in \( \mathcal{P}_2 \); \( X_1 \) (crosses), \( X_2 \) (stripes) and \( X_3 \) (dots) [Fig. 4(b)]. Suppose we construct a coordinated 3-coloring of \( \mathcal{P}_2 \) where three colors appear in each motif. We take \( L \leq K \), \( J \leq G \), such that \( |K: L| = 3 \) and \( |G: J| = 3 \). Observe that \( \text{Stab}_G(x) \subset L \) and \( \text{Stab}_G(x) \subset J \). Consider \( L = \{1\}_d; m_{010} \equiv m \text{dihedral group of order 2} \) where \( m_{010} \equiv m \) is a reflection with axis the line that passes through 0,0 as shown, and \( J = \{m_{010}\} \text{e}; \equiv \text{plm1} \) [where \( z(3, 0) \) is the translation with vector \((3, 0) \) and \( m_{010} \equiv m \) is a reflection with axis the vertical line that passes through 0,0] which satisfies \( J \cap K = \{1\}_d \). Using the formula in Corollary 5.6, we find that \( c = |L| = 2 \). Take the \( L \)-orbits of regions in \( M \) and \( \bar{L} \)-orbits of regions in \( zM \), \( z \notin J \) where \( \bar{L} = zk(Lz)^{-1} \) for some \( k \in K \). In particular, we let \( \bar{L} = zLz^{-1} \). These are colored black, white and gray in Fig.

![Figure 4](image-url)

(a) A frieze pattern \( \mathcal{P}_2 \) with global symmetry group \( G \equiv \text{plm1} \) and symmetry group of the motif \( g_M \) of \( G \) given by \( gK^{-1} \equiv 3m \). (b) The three \( G \)-orbits of regions: \( X_1 \) (crosses), \( X_2 \) (stripes) and \( X_3 \) (dots). (c) The \( L \)-orbits of regions in \( M \) and \( zLz^{-1} \)-orbits of regions in \( zM \). (d) The \( (G \cap L) \)-orbit representatives \( x_1, x_2 \) from the \( L \)-orbit of regions in \( M \) colored black, and the \( (G \cap zLz^{-1}) \)-orbit representatives \( x_3 \) from the \( zLz^{-1} \)-orbit of regions in \( zM \) colored white. (e) A coloring of \( \mathcal{P}_2 \).
4(c). Now, we construct \( T = T_1 \cup T \). We have \( T_1 = \{ x_1, x_2 \} \), a complete set of \((G \land L)\)-orbit representatives in the \( L\)-orbit of regions in \( M \) colored black, \( x_1 \in X_1 \), \( x_2 \in X_2 \) and \( T = \{ x_3 \} \) where \( x_3 \) is the \((G \land L)\)-orbit representative in the \( L\)-orbit of regions in \( zM \) colored white, \( x_3 \in X_1 \). The set \( T_2 = \{ x_3, x_4 \} \) is a complete set of \((G \land L)\)-orbit representatives in the \( L\)-orbit of regions in \( zM \) colored white [see \( x_1, x_2, x_3, x_4 \) in Fig. 4(d)]. A 3-coloring of \( P_0 \) is given by the partition \( P = JT \cup zJT \cup z^2JT \) where \( JT, zJT \) and \( z^2JT \) correspond, respectively, to \( P_1 \) (blue), \( P_2 \) (red) and \( P_3 \) (yellow).

Consider \( x_4 \in T_2 \setminus T \). Since \( x_4 \in T_2 \setminus T \), then \( x_4 \) has the color blue. Now \( x_4 \) is the image of \( x_2 \) under \( m_{0[0]} \frac{1}{2}, 0 \), and \( m_{0[0]} \frac{1}{2}, 0 \) fixes \( x_2 \). Since \( m_{0[0]} \frac{1}{2}, 0 \) permutes the colors then \( m_{0[0]} \frac{1}{2}, 0 \) sends \( T_1 \) (blue) also to \( (m_{0[0]} \frac{1}{2}, 0)T_1 \) (blue). We have \((m_{0[0]} \frac{1}{2}, 0)T_1 \cup T \) as the set consisting of regions colored blue in \((m_{0[0]} \frac{1}{2}, 0)M \) with \((m_{0[0]} \frac{1}{2}, 0)T_1 \cup T \supset \{ LT_1 \} \) so the coloring in \((m_{0[0]} \frac{1}{2}, 0)M \) is not perfect under its symmetry group \( zK^2 \). In fact, we only get a 2-coloring in each motif, despite the fact that \([K : L] = 3\), which is a contradiction.

Taking into consideration the characteristics of a coordinated coloring discussed previously, a method in arriving at a coordinated \( n\)-coloring of \( P \) where we have an \( n^*\)-coloring in a motif is given below.

(1) Determine the \( G\)-orbits of regions in \( P \). Suppose there are \( p \) number of \( G\)-orbits of regions in \( P \), namely \( X_1, X_2, \ldots, X_p \).

(2) Choose a motif \( M \) in \( P \) with symmetry group \( K \). Consider \( L \subseteq K \), \([K : L] = n^*\) satisfying \( \text{Stab}_K(x_i) \leq L \), \( x_i \in X_i \cap M \), \( i = 1, 2, \ldots, p \) and where \( c \) divides \( p \) and \( c \) is the number of \((G \land L)\)-orbits of regions in an \( L\)-orbit of regions in \( M \).

(3) Choose \( J \subseteq G \), \([G : J] = n \) satisfying \( J \cap K = G \cap L \) such that for every \( i \in \{1, 2, \ldots, p\} \) a conjugate of \( \text{Stab}_J(x_i) \leq J \), \( x_i \in X_i \).

(4) Form \( T = \{ x_1, x_2, \ldots, x_p \} \) the set of all \( J\)-orbit representatives \( x_i \) from \( X_i \) satisfying \( \text{Stab}_J(x_i) \leq J \), \( i = 1, 2, \ldots, p \).

(i) Suppose \( p = c \). Form \( T = \{ x_1, x_2, \ldots, x_p \} \) is a complete set of \((G \land L)\)-orbit representatives from an \( L\)-orbit of regions in \( M \).

(ii) Suppose \( p = cd \), \( d > 1 \). Form \( T = T_1 \cup T_2 \cup \ldots \cup T_d \) where \( T_1 = \{ x_1, x_2, \ldots, x_p \} \) is a set of \((G \land L)\)-orbit representatives from an \( L\)-orbit of regions in \( M \) and \( T_2 = \{ x_{1-q+1}, x_{1-q+2}, \ldots, x_c \} \), \( q = 2, 3, \ldots, d \), \( d \) is a set of \((G \land g_q \land g_q^{-1})\)-orbit representatives from \( g_q \land g_q^{-1} \)-orbit of regions in \( g_q \land g_q^{-1} \).

(5) Consider \( \{ g_1, g_2, \ldots, g_p \} \), a complete set of left \( \text{coset} \) representatives of \( J \) in \( G \). Form the partition \( P = \{ j \land JT \mid g \in G \} = \{ JT, j \land JT, \ldots, j \land JT \} \).

Let \( S = \{ C_1, C_2, \ldots, C_p \} \) be a set of \( n \) colors. To each \( g \land JT \) in \( P \), assign the color \( C_i \), \( i = 1, 2, \ldots, n \). Then \( P = \{ j \land JT \mid g \in G \} \) describes the coordinated \( n\)-coloring of \( P \).

Remark. In the above procedure we assume \( J \cap K = G \cap L \). It follows that \( J \cap gK \cap g^{-1} = G \cap L \) where \( g = (g_k) \land (g_k)^{-1} \) for some \( k \in H \). Here \( J = \text{Stab}_g(P_1) \) and \( L = \text{Stab}_g(P_1) \) in the \( n\)-coloring given by \( P = \{ P_1, P_2, \ldots, P_n \} \).

We first give an application of the aforementioned framework using step [4(i)]. Consider the frieze pattern \( P_1 \) in Fig. 2 and suppose each motif is divided into eight congruent triangles. We discuss the construction of a coordinated 4-coloring of the frieze pattern such that each motif is given four colors. There are two \( G\)-orbits of regions in \( P_1 \): \( X_1 \) (stripes) and \( X_2 \) (dotted) shown in Fig. 5(a). We pick \( L \) such that \([K : L] = 4\) and \( J \) such that \([G : J] = 4\). We have \( \text{Stab}_K(x_1) = \{ 1 \}_d \) and \( \text{Stab}_J(x_1) = \{ 1 \}_d \) for any \( x \in M \) and \( P_1 \), respectively. Any choice of subgroups \( L \) of \( K \) and \( J \) of \( G \) will satisfy \( \text{Stab}_K(x_1) \leq L \) and \( \text{Stab}_J(x_1) \leq L \). We choose \( L = \{ 1 \}_d \) and \( \text{Stab}_K(x_1) = \{ 1 \}_d \) is a reflection with axis the diagonal line passing through \( 0,0 \) as shown. We obtain \( G \cap L = \{ 1 \}_d \) and \( c = 0 \). We pick our \( G\)-orbit representatives from \( M \).

Now we consider \( J = \langle g_{1[0]}(1, 0) \rangle \approx p_{111}g \leq G \) where \( g_{1[0]}(1, 0) \) is a glide reflection with axis the horizontal line passing through \( 0,0 \) with glide vector \( (1, 0) \), \([G : J] = 4\) so that \( J \cap K = \{ 1 \}_d \) \( G \cap L \). We now obtain \( T \) as follows: first, we determine the four \( L\)-orbits of regions in \( M \): the regions colored black, gray, light gray and white [see Fig. 5(b)]. We form \( T = \{ x_1, x_2 \} \), the set consisting of \((G \land L)\)-orbit representatives from one particular \( L\)-orbit of regions where \( x_1 \in X_1 \) and \( x_2 \in X_2 \). In particular, we take \( x_1, x_2 \) from the \( L\)-orbit of regions in \( M \) colored black [Fig. 5(b)].

Let \( \{ 1 \}_d \approx p_{120, 2.0.0}, m_{0[1]} : m_{0[1]} \) be a complete set of left \( \text{coset} \) representatives of \( J \) in \( G \). Form the partition \( P = JT \cup 2JT \cup m_{0[0]}JT \cup m_{0[0]}JT \) and assign blue, red, yellow and green to \( JT, 2JT, m_{0[0]}JT \) and \( m_{0[0]}JT \), respectively. Then we obtain the coordinated 4-coloring of the frieze pattern shown in Fig. 5(d). Note that the coloring of \( M \) is described by \( \{ Lx \} \cup 2Lx \cup m_{0[1]}Lx \cup m_{0[0]}Lx \) where \( Lx_1, 2Lx_1, m_{0[1]}Lx_1 \) and \( m_{0[0]}Lx_1 \) are assigned the colors blue, red, yellow and green, respectively. In this example, we have \((J \cap K)T = (G \land L)T = Lx_1\).
As a second example, we explain how to arrive at the coordinated 4-coloring of the frieze pattern shown in Fig. 3, where each motif is given two colors using step (4)(ii). We start with the uncolored frieze pattern $P_1$ in Fig. 2, and assume each square motif is divided into four congruent isosceles triangles. There are two $G$-orbits of regions in $P_1$: $X_1$ (stripes) and $X_2$ (dotted) which we present in Fig. 6(a). We pick $G$-orbit representatives in $M$, namely $x_1^* \in X_1 \cap M$ and $x_2^* \in X_2 \cap M$. Note that we have $\text{Stab}_G(x_1^*) = \{1_{id}; m_{[01]} \} \cong m$ and $\text{Stab}_G(x_2^*) = \{1_{id}; m_{[01]} \} \cong m$, where $m_{[01]}$ are reflections with axes the vertical and horizontal lines passing through 0, respectively. Our choice of $L$ is $L = \{1_{id}; 2; m_{[01]}; m_{[01]} \} \cong 2mm$, where $[K : L] = 2$. This is an appropriate choice since $\text{Stab}_G(x_1^*) \leq L$ and $\text{Stab}_G(x_2^*) \leq L$. With this choice of $L$, we have $G \cap L = \{1_{id}; 2; m_{[01]}; m_{[01]} \} = L$ and $c = (4)(2)/(4)(2) = 1$ which divides $p = 2$. We choose $J = \{m_{[01]}; m_{[01]} \}, z^{(4, 0)} \cong 2pmm$. $[G : J] = 4$ (to obtain a 4-coloring) satisfying $J \cap K = G \cap L$. We confirm the suitability of the subgroup $J$ by noting that $\text{Stab}_G(x_2^*) = \{1_{id}; m_{[01]} \} \leq J$ and $\text{Stab}_G(x_1^*) = \{1_{id}; m_{[01]} \} \leq J$.

Next, we form the set $T$. Since $c = 1$ and $p = 2$, then we obtain $T$ by selecting one $G$-orbit representative from each of the motifs $M$ and $z^2 M$, $z^2 (2, 0) \notin J$. We first consider the $L$-orbits of regions in $M$ and $(z^2 L z^{-2})$-orbits of regions in $z^2 M$ [Fig. 6(b)]. Then we form $T = T_1 \cup T_2$ with $T_1 = \{x_1, x_2 \}$, where $x_1 = x_1^* \in X_1$, a $(G \cap L)$-orbit representative from the $L$-orbit of regions in $M$ colored black and $T_2 = \{x_3, x_4 \}$, a $(G \cap z^2 L z^{-2})$-orbit representative from the $(z^2 L z^{-2})$-orbit of regions in $z^2 M$ colored gray [Fig. 6(c)].

Consider $\{1_{id}; z^{(1, 0)}; z^{(2, 0)}; z^{(3, 0)} \}$, a complete set of left coset representatives of $J$ in $G$. The coordinated 4-coloring of $P$ in Fig. 3 is determined by the partition $P = JT \cup zJT \cup z^2JT \cup z^3JT$ where the colors blue, yellow, red and green are assigned to $JT$, $zJT$, $z^2JT$ and $z^3JT$, respectively. In this coloring of $P$, each motif is given two colors. The coloring of $M$, for example, is described by $Q = Lx_1 \cup 4^* L x_1$ where $L x_1$ and $4^* L x_1$ are given the colors blue and red, respectively. We have $L x_1 = (G \cap L) T_1 = (J \cap K) T_1$.

The next theorem describes the situation where elements of $G \cap gKg^{-1}$ (symmetries) act transitively on the colors in the motif $gM(g \in G)$.

**Theorem 5.8.** Suppose the assumptions of Theorem 5.4 hold. Assume the number $c$ of ($G \cap L$)-orbits of regions in $\overline{Lm}$ where $m \in P_1 \cap gM(g \in G)$ and $\overline{L} = \text{Stab}_{gKg^{-1}}(P_1)$ is equal to $p$, the number of $G$-orbits of regions in $P$. Then $G \cap gKg^{-1}$ acts transitively on the colors used in coloring $gM$.

**Proof.** Suppose $T^*$ consists of a complete set of $(G \cap L)$-orbit representatives in $\overline{Lm}$ where $m \in P_1 \cap gM(g \in G)$ and $\overline{L} = \text{Stab}_{gKg^{-1}}(P_1)$. Assume $|T^*| = c$. Since $c = p$ then $T^*$ consists of a complete set of $(G \cap gKg^{-1})$-orbit representatives in $gM$ or, equivalently, this means $G \cap gKg^{-1}$ forms $p$ orbits of regions in $gM$. Since $\text{Stab}_{gKg^{-1}}(P_1) = \overline{L}$ then $\text{Stab}_{gKg^{-1}}(P_1) = G \cap \overline{L} \leq G \cap gKg^{-1}$. Then the partition $Q^* = \{k(G \cap \overline{L}) T^* | k \in (G \cap gKg^{-1})\}$ where $T^* = \{t_1, t_2, \ldots, t_r\}$ is a set of $(G \cap gKg^{-1})$-orbit representatives from each $X^*_r$ such that $\text{Stab}_{G \cap gKg^{-1}}(X^*_r) \leq (G \cap \overline{L})$ is a coloring of $gM$ for which elements of $G \cap gKg^{-1}$ permute the colors and $G \cap gKg^{-1}$ is transitive on the colors.

For the coordinated 4-coloring shown in Fig. 5(d), $G \cap K = \{1_{id}; 2; m_{[01]}; m_{[01]} \} \cong 2mm$ permutes the colors in $S_1$ and $G \cap K$ is transitive on the four colors in $S_2$. For any two colors, there is an element of $G \cap K$ that sends one color to the other. In this example we have $G \cap L = \{1_{id}\}$ where $[G \cap K : G \cap L] = 4$. The coloring of $M$ is described by the partition $Q = \{k(G \cap L) T | k \in K\} = T \cup 2T \cup m_{[01]} T \cup m_{[01]} T$ where $\{1_{id}; 2; m_{[01]}; m_{[01]} \}$ is a complete set of left coset representatives of $(G \cap L)$ in $(G \cap K)$ and $T = \{x_1, x_2\}$ is the set consisting of $(G \cap L)$-orbit representatives in one $L$-orbit of regions in $M$ which also happen to be a complete set of $(G \cap K)$-orbit representatives in $M$. It can also be verified in a similar manner that $G \cap gKg^{-1}$ permutes the colors and is transitive on the four colors in $gM$.

In Fig. 7, we show a coordinated 3-coloring of $P_2$ where $G \cap K$ does not act transitively on the colors used in $M$. In this situation, $G \cap K = \{1_{id}; m_{[01]} \} \cong m$. Observe that $m_{[01]}$ does not send a region colored blue to a region colored yellow in $M$. Thus, $G \cap K$ does not act transitively on the colors in $M$. Take blue $(P_1)$, a color in $M$. We have $\text{Stab}_E(P_1) = L = \{1_{id}; m_{[01]}\}$. Since $(G \cap L) = L$ then there is only one $(G \cap L)$-orbit of regions in $M$, so $c \neq p$.

6. Examples of coordinated colorings of planar patterns

The following examples show coordinated colorings of planar patterns. The first example presents the construction of coor-
ordinated 4-colorings of a planar pattern with global symmetry group a plane crystallographic group \( p4mm \) having local symmetries that include fivefold rotational symmetries that are not global symmetries. The second example shows coordinated 6-colorings of a planar pattern with global symmetry group a plane crystallographic group \( p6mm \) having local symmetries that include fourfold rotational symmetries that are not global symmetries.

Consider a planar symmetrical pattern \( \mathcal{P}_3 \) consisting of disjoint congruent copies of the regular pentagon \( M \) that is divided into ten congruent regions [Fig. 8(a)]. The global symmetry group of \( \mathcal{P}_3 \) is \( G = \langle 4^+ 0 0; m_{110} 0 0; z_1(1, 0); z_2(0, 1) \rangle \cong p4mm \) where \( 4^+ \) is the 90° counterclockwise rotation about the coordinates 0,0, the center of \( M \), \( m_{110} \) is the reflection with axis the line passing through 0,0 as shown, and \( z_1(1, 0), z_2(0, 1) \) are two linearly independent translations with respective vectors \((1, 0), (0, 1)\). The symmetry group of \( M \) is \( K = \langle 5^+ \frac{1}{5}; m_{111} \rangle \cong 5m \) (dihedral group of order 10) where \( 5^+ \frac{1}{5} \) is the 72° counterclockwise rotation about the coordinates \( \frac{1}{5}, \frac{1}{5} \), the center of \( M \), and \( m_{111} \) is the reflection with axis the line passing through 0,0 as shown. We have \( \text{Stab}_G(x) = \{1id\} \) and \( \text{Stab}_G(x^*) = \{1id\} \) for any region \( x, x^* \) in \( M \) and \( \mathcal{P}_3 \), respectively, so any choice of subgroups \( L \) of \( J \) and \( K \) of \( G \) will satisfy \( \text{Stab}_K(x) \leq L \) and \( \text{Stab}_K(x^*) \leq J \). The group \( K \cong 5m \) has subgroups of index 1, 2, 5 and 10. Since we want a 4-coloring of \( \mathcal{P}_3 \), then the only choice for \( L \) is either a subgroup of index 1 or 2 in \( K \). If \( L = K \), we obtain a trivial coloring of each motif. To show non-trivial colorings of each motif we pick \( L = \langle 5^+ \rangle \cong 5 \) (cyclic group of order 5), the only subgroup of index 2 in \( K \). Observe that \( G \cap L = \{1id\} \). The list of subgroups \( J_i \) with \( [G : J_i] = 4, i = 1, 2, \ldots, 14 \) satisfying \( J_i \cap K = G \cap L = \{1id\} \) is presented in Table 1. These subgroups will yield coordinated 4-colorings of \( \mathcal{P}_3 \) using the partition given, where the colors blue, yellow, red and green are assigned, respectively, to \( J_1, g_3 J_1, g_3 \), \( J_2, J_2, \) \( \{1id, g_2, g_3, g_4\} \) a complete set of coset representatives of \( J_1 \) in \( G \). The subgroups have been obtained with the aid of the software Groups, Algorithms and Programs (GAP) (The GAP Group, 2013).

There are five \( G \)-orbits of regions in \( \mathcal{P}_3 \); \( X_1 \) (crosses), \( X_2 \) (gray dotted), \( X_4 \) (black dotted), \( X_4 \) (stripes) and \( X_5 \) (asterisks) [see Figs. 8(b) and Fig. 8(c) for the \( G \)-orbits of regions in \( \mathcal{P}_3 \) and \( M \), respectively]. Now from Corollary 5.6, \( c = |L| = 5 \) and so \( c = p \), the number of \( G \)-orbits of regions in \( \mathcal{P}_3 \). Thus the complete set of \( G \)-orbit representatives will be formed by determining \( (G \cap L) \)-orbit representatives from an \( L \)-orbit of regions in \( M \). We form the two \( L \)-orbits of regions in \( M \), regions colored black and gray [see Fig. 8(d)].

For the first set of colorings we use \( T = \{x_1, x_2, \ldots, x_5\} \) consisting of \((G \cap L)\)-orbit representatives from one particular \( L \)-orbit of regions colored black [Fig. 8(e)]. Consider for example the subgroup \( J_1 = \{m_{110} 0 0; z_1(1, 0); z_2(0, 1) \} \cong p1m1 \). To color \( \mathcal{P}_3 \), we form the partition \( P = J_1 T \cup 4^+ J_1 T \cup 2 J_1 T \cup 4^+ J_1 T \) where \( \{1id, 4^+ 0 0; 2 0 0, 4^- 0 0\} \) is a complete set of left coset representatives of \( J_1 \) in \( G \). We assign the respective colors blue, yellow, red and green to \( J_1 T, 4^+ J_1 T, 2 J_1 T \) and \( 4^+ J_1 T \) resulting in the coordinated 4-coloring shown in Fig. 9(1). To get the respective colorings in Fig. 9(2)–(14) we use \( J_i, i = 2, 3, \ldots, 14 \) and \( T = \{x_1, x_2, \ldots, x_5\} \) in the given partition in Table 1, column 3.
For the second set of colorings we get \((G \cap L)\)-orbit representatives from the \(L\)-orbit of regions colored gray [Fig. 8(e)] and form \(T' = \{x_1, x_2, x_3, x_4, x_5\}\). Using the subgroups \(J_i\) from Table 1 and replacing \(T\) in the given partition with \(T'\) we get the other coordinated 4-colorings of \(P_3\), which are shown in Fig. 9(15)–(28). For example, using the subgroup \(J_1 = (m_{140} \colon z_1(1, 0); z_2(0, 1)) \cong p_{1m1}\) of \(G\), we obtain another coordinated 4-coloring of \(P_3\), shown in Fig. 9(15), given by the partition \(J_1 T' = \{4^+ J_1 T', 2J_1 T', 4^+ J_1 T', 2J_1 T'\}\) where \(J_1 T', 4^+ J_1 T', 2J_1 T', 4^+ J_1 T'\) correspond, respectively, to blue, yellow, red and green.

For each of the coordinated 4-colorings of \(P_3\) shown in Fig. 9, the group \(G \cap gKg^{-1}\) acts transitively on the colors of \(gM\).

Some pairs of colorings in Fig. 9 are equivalent colorings. Roth (1982) defines two colorings of a symmetrical pattern as being equivalent if one of the colorings is obtained from the other by (i) a bijection from \(S_1\) (set of colors used in the first coloring) to \(S_1\) (set of colors used in the second coloring), or (ii) a symmetry in the symmetry group \(G\) of the symmetrical pattern, or (iii) a combination of (i) and (ii). In Fig. 9, the pairs of colorings #2 and #16, #7 and #21, #8 and #22, #9 and #23, #11 and #25, #12 and #26 are equivalent. The second coloring in any given pair can be obtained from the first by applying the reflection \(m_{137}\) with axis the diagonal line passing through 0.0 as shown.

We now illustrate a 6-coloring of the planar symmetrical pattern \(P_4\) given in Fig. 10(a) consisting of disjoint congruent copies of a motif, the symmetrical figure, which is divided into four congruent regions. The symmetry group of \(P_4\) is \(G = \{6^0, 0, 0; m_{140}, 0, 0; z_1(1, 0); z_2(0, 1)\} \cong p_{6mm}\) where \(6^0\) is the \(60^\circ\) counter-clockwise rotation about the coordinates 0, 0, \(m_{140}\) is the reflection with axis the horizontal line passing through 0.0, and \(z_1(1, 0), z_2(0, 1)\) are two linearly independent
translations with respective vectors \((1, 0), (0, 1)\). The symmetry group of \(K = \{4^+ J_1 T', 0; m_{140}\} \cong 4mm\) where \(4^+\) is the \(90^\circ\) counter-clockwise rotation with center at the coordinates \(0.5, 0\), the center of \(M\).

Fig. 10(b) shows two \(G\)-orbits of regions in \(P_2\), namely \(x_1\) and \(x_2\) (dotted). We pick \(G\)-orbit representatives in \(M\) of the symmetrical pattern, namely \(x_1^* \in X_1 \cap M\) and \(x_2^* \in X_2 \cap M\) [Fig. 10(c)]. We obtain \(\text{Stab}_G(x_1^*) = \{1_{4d}; m_{140}\} \cong m\) and \(\text{Stab}_G(x_2^*) = \{1_{4d}; m_{141}\} \cong m\) where \(m_{141}\) is the reflection with axis the vertical line passing through \(0.5, 0\). We first pick \(L = \{K\}\). Note that \(K \cong 4mm\) and has subgroups of index 1, 2, 4 and 8. Since we want a 6-coloring of \(P_4\) where the coloring of each motif is non-trivial, then the only choice for \(L\) are the subgroups of index \(2\) and \(4\). Since we want \(\text{Stab}_G(x_1) \leq L\) and \(\text{Stab}_G(x_2) \leq L\), then the subgroup \(L = \{1_{4d}; 2 \frac{1}{2} J_1 T', m_{140} \cong 4mm\} \cong m\) where \([K : L] = 2\) is the only suitable choice. In \(L, 2 \frac{1}{2} J_1 T'\) is a \(180^\circ\) rotation about the coordinates \(0.5, 0\).

We now consider subgroups \(L = 4\) of index 6. From the list of subgroups of index 6 in \(G\) generated by \(GAP\), the only possible choice for \(J\) is \(J = \{2, 0, 0; m_{140}, 0, 0; z_1(1, 0); z_2(0, 2)\} \cong p_{2mm}\) since this satisfies \(J \cap K = G \cap L = \{1_{4d}; 2 \frac{1}{2} J_1 T', m_{140} \cong 4mm\}\). Note that \(G \cap L = L\) so
\[
\epsilon = \frac{|L|\cdot|\text{Stab}_G(x_1^*)|}{|G| \cdot |\text{Stab}_G(x_1)|} = \frac{(4)(2)}{(4)(2)} = 1.
\]

Now we form the \(L\)-orbit of regions in \(M\) and \(z_2Lz_2^{-1}\)-orbit of regions in \(z_2M\), \(z_2(0, 1) \notin J\) [Fig. 10(d)], the \(G\)-orbit of regions in \(M\) and \(z_2M\) are shown in Fig. 10(c). Since \(c = 1\) and \(p = 2\) we choose one \(G\)-orbit representative from each motif \(M\) and \(z_2M\) to make up \(T\). We form \(T = T_1 \cup T_2\) where \(T_1 = \{x_1\}\), \(x_1\) is the \((G \cap L)\)-orbit representative from the \(L\)-orbit of regions in \(M\) colored black, and \(T_2 = \{x_2\}\), \(x_2\) is the \((G \cap z_2Lz_2^{-1})\)-orbit representative from the \(z_2Lz_2^{-1}\)-orbit of regions in \(z_2M\) colored gray [Fig. 10(e)]. We infer that the
Figure 9: Coordinated 4-colorings of $P_3$, where each motif is given two colors.
choice of \( J \) is suitable since \( \text{Stab}_G(x_1) = \{1_{44}; m_{110}0,0\} \leq J \) and \( \text{Stab}_G(x_2) = \{m_{110}0,0,0\} \leq J \). Consider \( \{1_{44}; 6^+0,0; 3^+0,0; z_2(0,1); 6^+ \overline{T}_0; 3^+ \overline{T}_2\} \), a complete set of left coset representatives of \( J \) in \( G \); \( 6^+0,0 \) and \( 6^+ \overline{T}_0 \) are 60\(^\circ\) counterclockwise rotations about the coordinates 0,0 and \( \overline{T}_0 \), respectively, and \( z_2(0,1) \) is a translation with vector (0, 1). Form the partition \( P = JT \cup (6^+0,0)JT \cup (3^+0,0)JT \cup z_2JT \cup (6^+ \overline{T}_0)JT \cup (3^+ \overline{T}_2)JT \) where the colors blue, yellow, red, green, pink and violet are assigned to the sets \( JT \), \( (6^+0,0)JT \), \( (3^+0,0)JT \), \( z_2JT \), \( (6^+ \overline{T}_0)JT \) and \( (3^+ \overline{T}_2)JT \), respectively. We obtain the coordinated 6-coloring shown in Fig. 10(f).

Suppose we use another set \( T' \) of \( G \)-orbit representatives still coming from each motif \( M \) and \( z_2M \). We form \( T' = T'_1 \cup T'_2 \) where \( T'_1 = \{x'_1\} \) and \( T'_2 \) is the \( (G \cap L) \)-orbit representative from the \( L \)-orbit of regions in \( M \) colored gray, and \( T'_2 = \{x'_2\} \). The \( (G \cap z_2Lz_2^{-1}) \)-orbit representative \( x'_2 \) is the \( (G \cap z_2Lz_2^{-1}) \)-orbit of regions in \( z_2M \) colored black [Fig. 10(e)]. The same choice of \( J \) = \( \{20,0; m_{110}0,0; z_2(1,0); z_2^2(0,2)\} \) is suitable since \( \text{Stab}_G(x'_1) = \{1_{44}; m_{110}0,0,0\} \leq J \) and \( \text{Stab}_G(x'_2) = \{m_{110}0,0,1\} \leq J \). Consider \( \{1_{44}; 6^+0,0; 3^+0,0; z_2(0,1); 6^+ \overline{T}_0; 3^+ \overline{T}_2\} \) a complete set of left coset representatives of \( J \) in \( G \). Replacing \( T \) with \( T' \) in the partition \( P \) above, and using the same assignment of colors, we obtain the coordinated 6-coloring shown in Fig. 10(g).

The two coordinated 6-colorings of \( P \) are equivalent colorings. Observe that if we replace the respective colors blue, yellow, red, green, pink and violet in the coloring in Fig. 10(f) with the colors green, pink, violet, blue, yellow and red we obtain the coloring in Fig. 10(g).

7. Chromatic, partial chromatic and achromatic operations
In this section we discuss chromatic, partially chromatic or achromatic properties of symmetries or partial operations of a coordinated coloring of a symmetrical pattern. Given a...
symmetrical pattern $\mathcal{P}$ consisting of disjoint congruent motifs, an isometry of $E^2$ that sends one motif to another we will call a partial operation on $\mathcal{P}$ (Sadanaga et al., 1980). This isometry may not be a global or local symmetry of $\mathcal{P}$. The set of partial operations sending a motif to another form a chromatic groupoid, with the operation inherited from $\text{Isom}(E^2)$.

In a coordinated coloring of $\mathcal{P}$, a global symmetry, a local symmetry or a partial operation can have the property that it is chromatic (it exchanges or moves a number of colors without leaving fixed any color), partially chromatic (it leaves fixed one or more colors but not all colors) or achromatic (all colors are left fixed).

If the coloring of $\mathcal{P}$ or its motif $gM$ has two colors then $G$ or $gKg^{-1}$, respectively, is a dichromatic symmetry group. Whenever the movement of a symmetry or partial operation is accompanied by a change in color then we add to its corresponding symbol a ‘prime’ ($'$). For a translation, the direction in which the change of colors occurs is indicated where necessary by a suffix.

For a $n$-coloring of $\mathcal{P}$ or $gM$ where $n > 2$, then $G$ or $gKg^{-1}$, respectively, is a polychromatic symmetry group. In polychromatic symmetry groups, a symmetry or operation may move $n_1$ colors one step along a sequence, or two steps, or more. Its corresponding symbol is then denoted with a superscript $(n_1)$. This is the case when we have the transformation sending color $C_1$ to $C_2$, $C_2$ to $C_3$, ..., $C_{n_1}$ back to $C_1$. A symmetry or operation that has partial chromaticity will be denoted with a superscript $(n_1, n_2)$, where the symmetry or operation moves $n_1$ colors and fixes $n_2$ colors. Here $n_1, n_2 \leq n$. An outer superscript $(n)$ will be used to denote the number of colors in the $n$-coloring. The notations for dichromatic and polychromatic symmetry groups are adapted from the work of Van der Warden & Burchardt (1961).

Even if coordinated colorings are perfect colorings where the motifs are perfectly colored as well, it can be shown in the discussion that follows that these can have different chromaticity properties.

First, we consider the coordinated coloring of $\mathcal{P}_1$ given in Fig. 3 which has the following properties. The local chromatic symmetries of $M$ exchanging blue and red are $4^*$, $4^-$, $m^*_2$, and $m^*_2$. On the other hand, the local achromatic symmetries of $M$ fixing the colors blue and red are $2$, $m_{[10]}$ and $m_{[01]}$. The dichromatic group of $M$ is $4\, mm'$: we have a $90^\circ$ rotation and two reflections that change colors, while two reflections fix colors. The dichromatic group of each of the other motifs in $\mathcal{P}_1$ is also $4\, mm'$.

Global symmetries in $\mathcal{P}_1$ can be characterized as: $z(0)^*(1, 0)$ (chromatic) sending blue to yellow, yellow to red, red to green and green to blue; $2^{(2,2)}$, $0$ and $m^{(2,2)}$, $0$ (partially chromatic) exchanging yellow and green, and fixing the colors red and blue; and $m_{[10]}$ (achromatic) fixing all the colors. The 4-coloring of the frieve pattern is described by the group $[p^{(4,1}\, 2^{(2,2)}\, m^{(2,2)}\, m^*]^{(4)}$.

The chromatic partial operations that map $M$ to $zM$ are characterized as follows (see Fig. 11). Sending blue to yellow as well as red to green are $z(0)^*(1, 0)$, $2^{(2)}$, $0$, $m^{(2)}$, $0$ and $g_{[01]}(1, 0)$ [glide reflection with axis the horizontal line passing through 0,0 and with glide vector $(1, 0)$]. On the other hand, sending blue to green and red to yellow are $4^{(2)}(1, 1)$, $2^{(2)}$, $1$, $2^{(2)}$, $0$ and $g_{[10]}(1, 1)$ [glide reflection with axis the line passing through $1,0$ and with glide vector $(1, 1)$].

On the other hand, the chromatic partial operations that map $M$ to $z^3M$ interchanging red and green; exchanging blue and yellow, as well as yellow to red; 2 $m^{(2)}$, $1$, $0$, $g_{[01]}(1, 0)$, $m^{(2)}$, $0$ (Fig. 11).

The achromatic partial operations are $4^+, 4^-$, $1, 0$, $g_{[01]}(1, 1)$ and $g_{[10]}(1, 1)$, $1, 0$ (Fig. 11).

For the chromatic partial operations that map $M$ to $z^3M$ we have the following (Fig. 11): the partial operations $(z^3)^*(2, 0)$, $2^{(2)}$, $0$, $m^{(2)}$, $0$ and $g_{[01]}(2, 0)$ sending blue to green, and red to yellow; and the partial operations $4^{(2)}$, $2^{(2)}$, $1$, $0$, $g_{[10]}(1, 1)$, $0$ and $g_{[01]}(1, 1)$, $0$ sending blue to yellow, and red to green.

We can characterize the other partial operations that map $M$ and $z^3M$, where $a$ is an integer, in a similar manner as above.

Note that among the generators of the group $[p^{(4,1}\, 2^{(2,2)}\, m^{(2,2)}\, m^*]^{(4)}$ of the coloring shown in Figs. 3 and 11, only the translation has a chromatic property. The partial operations, however, as discussed above are achromatic. This is an example where partial operations that make up a groupoid have interesting chromaticity properties.

As a second example we consider the coordinated coloring of $\mathcal{P}_1$ in Fig. 5. Each motif has polychromatic group $[4^*(4)\, m^{(2)}\, m^{(2)}]^{(4)}$. Chromatic local symmetries of $M$ are $4^*(4)$ and $4^*(4)$, each moving four colors in a cycle: $m^{(2)}$, $0$ exchanging blue and yellow, as well as red and green; $m^{(2)}$, $0$ exchanging blue and green, and also yellow and red. The partial achromatic local symmetries include $m^{(2)}$, $0$ fixing blue and red and exchanging yellow and green, and $m^{(2)}$, $0$ fixing yellow and green and exchanging blue and red.

The coloring of the frieze pattern in Fig. 5 is described by the group $[p^{(4)}\, 2^{(2,2)}\, m^{(2)}\, m^*]^{(4)}$. Chromatic global symmetries are characterized as follows, exchanging pairs of colors: $z(2)^*(1, 0)$ exchanging blue and yellow, as well as yellow to red; $2^{(2)}$, $0$ exchanging blue and red, as well as yellow and green; $m^{(2)}$, $0$ exchanging blue and green, as well as yellow and red; and $m^{(2)}$, $0$ exchanging blue and yellow, as well as green and red. The coloring also has achromatic global symmetries. Examples...
include the translation \( z^2(2, 0) \) and the glide reflection \( g_{[10]}(1, 0) \).

Here we give one set of chromatic partial operations that map \( M \) to \( \mathcal{z}M \) (see Fig. 12). These are: \( z^{(2)}(1, 0) \) exchanging blue and yellow, as well as green and red; \( 2^{(2)} \frac{1}{2}, 0 \) exchanging blue and green, as well as red and yellow; \( m_{[01]}^{(2)} \frac{1}{2}, 0 \) exchanging blue and red, as well as yellow and green; \( g_{[01]}^{(4)}(1, 0) \frac{1}{2}, 0 \) sending blue to yellow, yellow to red, red to green and green to blue; and \( g_{[01]}^{(4)}(1, 0) \frac{1}{2}, 0 \) sending blue to green, green to red, red to yellow and yellow to blue. The partially achromatic operations are described as follows: \( 4^{(2,2)+} \frac{1}{2}, 0 \) fixing green and yellow, exchanging red and blue; \( 4^{(2,2)-} \frac{1}{2}, 0 \) fixing blue and red, exchanging yellow and green. The glide reflection \( g_{[01]}(1, 0) \) fixes all colors.

Finally, we consider the coordinated 4-colorings of \( \mathcal{P}_3 \) presented in Fig. 9. These are all perfect colorings but they have different chromaticity properties. The colorings can be described by one of the following polychromatic groups: 

\[
[\mathfrak{p}^4 m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^4 m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)}, [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)}.
\]

To illustrate, the coordinated 4-coloring of \( \mathcal{P}_3 \) in Fig. 9(1) is described by the group \( [\mathfrak{p}^4 m^{(2,2)} m^{(2,2)}]^{(4)} \). All translation symmetries fix the colors. The group elements include \( 4^{(2,2)+} 0,0 \) (chromatic) sending blue to yellow, yellow to red, red to green, and green to blue, \( m_{[01]}^{(2)} 0,0 \) (partially chromatic) fixing yellow and green, exchanging blue and red, and \( m_{[11]}^{(2)} 0,0 \) (partially chromatic) exchanging yellow and blue, as well as red and green.

On the other hand, the coordinated coloring in Fig. 9(2) is described by the group \( [\mathfrak{p}^4 m^{(2,2)} m^{(2,2)}]^{(4)} \). As in the coloring described in the preceding paragraph, all translation symmetries fix the colors. The reflection \( m_{[01]}^{(2)} 0,0 \) is chromatic, exchanging blue and red, as well as yellow and green, and the reflection \( m_{[11]}^{(2)} 0,0 \) is also chromatic, exchanging blue and green, as well as yellow and red. The rotation \( 4^{(2,2)-} 0,0 \) is also chromatic, exchanging blue and yellow, as well as red and green.

Consider the coordinated coloring in Fig. 9(6). It is described by the polychromatic group \( [\mathfrak{p}^{2,2} 4^{(2,2)} m^{(2,2)} m^{(2,2)}]^{(4)} \). The translation \( z_{[01]}(0, 1) \) (partially chromatic) fixes green and yellow, and exchanges blue and red. The polychromatic group has suffix \( [\mathfrak{p}^{2,2}] \) where the subscript \([01]\) denotes the direction of the translation in consideration. The global symmetries \( 4^{(2,2)+} 0,0 \) (chromatic) send blue to yellow, yellow to red, red to green, and green to blue; \( m_{[01]}^{(2)} 0,0 \) (partially chromatic) fixes yellow and green, and exchanges blue and red; and \( m_{[11]}^{(2)} 0,0 \) (chromatic) exchanges blue and green, as well as yellow and red.

The polychromatic groups of the other colorings can be described in the same manner as the groups discussed above.

Every motif in each coordinated coloring has dichromatic edge group \( 5m' \), that is, in a 2-coloring of a motif the fivefold rotational symmetry fixes the colors and the reflections exchange the two colors.

8. Conclusions and recommendation

The main contribution of this study to the existing theories of color symmetry is the development of a systematic method to arrive at coordinated colorings of symmetrical patterns. To arrive at a coordinated coloring of a symmetrical pattern \( \mathcal{P} \) consisting of disjoint congruent symmetric motifs, we take into account the global and local color symmetries of \( \mathcal{P} \), the transitive action of the global symmetry \( G \) on the set of colors used in the overall pattern, and the transitive action of the symmetry group \( gKg^{-1} \) of the motif on its colors. Moreover, for the elements of \( G \) that act as global and local color symmetries, a consistency of the color permutations is required. The development of the framework for a coordinated coloring required an understanding of the concepts of perfect colorings and their transitivity properties, both on the global and local scale. The results of our study point out that obtaining a coordinated coloring relies on orbit and stabilizer conditions that come with the right choices of the (i) subgroup \( J \) of \( G \) that will result in a partition describing the perfect coloring of the symmetrical pattern, (ii) subgroup \( \mathcal{L} \) of \( gKg^{-1} \) that will give a perfect coloring of the motif \( gM \) and (iii) relation of \( J \) and \( \mathcal{L} \) to ensure coordination between the color symmetries of the colorings described in (i) and (ii). As pointed out by Senecal (1988) in the well-written paper on color symmetry, the problem of the classification of repeating colored symmetrical patterns is a subgroup problem, as much work entails the knowledge of the subgroup structure of plane crystallographic groups and frieze groups, and the relations of these subgroups. This work points to this fact as well.

As a next step in the study, we would like to propose a method for obtaining coordinated colorings of a symmetrical pattern \( \mathcal{P} \) containing more than one transitivity class of motifs under the symmetry group of \( \mathcal{P} \). Consequently, there will be more than one local symmetry group type in \( \mathcal{P} \). Another possible extension of this work would be to look at coordinated colorings of non-Euclidean patterns and three-dimensional symmetrical structures.

We also propose an in-depth and detailed characterization of chromatic groupoids in the study of coordinated colorings.

Another aspect that could be studied in relation to coordinated colorings is the determination of a list of types of three- and two-dimensional crystallographic groups and frieze groups that act as a global symmetry group, together with the possible local symmetry group types of the motifs, including
non-identity stabilizer subgroups of the fundamental designs. For example, in the frieze pattern \( P_1 \) of Fig. 3, we have \( G \cong p2mm \), \( L \cong 2mm \) and \( \text{Stab}_G(x) \cong m \). In the plane pattern \( P_4 \), we have \( G \cong p6mm \), \( L \cong 2mm \) and \( \text{Stab}_G(x) \cong m \) where \( x \) is a region in the pattern.

Another interesting object of study will be to look at global and local color symmetries in cultural artifacts, following studies on color symmetries in culture in an area called Sulu (see e.g. De Las Peñas et al., 2018). It is suggested one starts with cultural artworks that are rich in global and local symmetries. The scarf from the Tausugs of Jolo, Sulu (De Las Peñas et al., 2018) is an example of a cultural artwork that depicts the presence of global and local symmetries and color symmetries. In fact, Grünbaum (2002) talks about various levels of orderliness, and global and local symmetry. He proposes that the study of different kinds of orderly structures may lead to a better understanding of designs and art in various ancient and modern cultures.

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