2014

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Muga, F.P., II. (2014). On the maximum leaf number of a family of circulant graphs. Lecture Notes in Engineering and Computer Science. Volume 1, 2014, Pages 46-49 World Congress on Engineering and Computer Science 2014, WCECS 2014; San Francisco; United States; 22 October 2014 through 24 October 2014; Code 109530.

On The Maximum Leaf Number of a Family of Circulant Graphs

Felix P. Muga II

***Abstract*—This paper determines the maximum leaf number and the connected domination number of some undirected and connected circulant networks which are optimal among all the maximum leaf numbers and connected domination numbers of circulant networks of the same order n and the same degree $2k$. We shall tackle this problem by working on the largest possible number of vertices between two consecutive jump sizes.**

***Index Terms*—leaf number, connected domination number.**

I. Introduction

Let G be a simple connected graph. The spanning tree of G is its subgraph that contains all its vertices and has no cycles. A maximum leaf spanning tree (MLST) of G has the most possible number of leaf vertices among all the spanning trees of G.

The number of leaf vertices of a MLST of G denoted by $\ell(G)$ is the *maximum leaf number* of G.

The problem of finding a maximum leaf spanning tree of G is the MLST problem.

A connected dominating set of G is a subset D of the vertex set of G that induces a connected subgraph of G such that every vertex in G is either in D or adjacent to a vertex in D. The minimum connected dominating set (MCDT) has the smallest possible number of vertices among all connected dominating sets of G.

The number of elements of a MCDT denoted by $d(G)$ is the *minimum connected domination number* of G.

R. J. Douglas [3] showed that $d(G) + \ell(G)$ is the order of G.

Hence, the problem of finding the MCDT of G is equivalent to the MLST problem.

II. Leaf Number of a Family of Circulant Networks

Let N and k be integers such that $N \geq 3$ and $k \geq 1$.

Consider a connected and undirected circulant network $G = C\big(n; \pm(s_1, s_2, \ldots, s_k)\big)$ of order N and degree $2k$ such that $N \geq 2k+1$.

The vertices s_i and $N - s_i$, for all $i = 1, 2, \ldots, k$ are called the jump sizes of G. The jump sizes under consideration are ordered such that $1 = s_1 < s_2 < \ldots < s_k < \dfrac{N}{2}$.

Let the vertices of G be labelled as $0, 1, \ldots, N-1$.

Suppose S is the ordered list that contains all the jump sizes of G. Then we can also write G as $C(N; S)$.

We shall find $d(G_i)$ and $\ell(G_i)$ for $i = 1, 2$ where

1) $G_1 = C\big(N; \pm(iq+1)\big)$, $\forall i = 0, 1, \ldots, k-1$, $q \geq 1$, and $N = (2k-1)q + 2$.
2) $G_2 = C\big(n; \pm(s_1, s_2)\big)$ where $N = (2k-1)q + r + 2$, $s_1 = iq + 1$, and $s_2 = (k - r_1 + j)q + j + 2$, $\forall i = 0, 1, \ldots, k - r_1 - 1$, $\forall j = 0, 1, \ldots, r_1 - 1$, $q \geq 1$, and $r = 1, 2, \ldots, 2k-2$ with $r_1 = \left\lfloor \dfrac{r}{2} \right\rfloor$.

Note that if $r = 1$, then $r_1 = 0$ and the jump of sizes of G_2 are similar to those of G_1.

We shall show that

$$\mathcal{M}\big(n,\ 2k\big) = \begin{cases} d(G_1) & \text{if } r = 0 \\ d(G_2) & \text{if } r > 0 \end{cases}$$

$$\mathcal{L}(n, 2k) = \begin{cases} \ell(G_1) & \text{if } r = 0 \\ \ell(G_2) & \text{if } r > 0 \end{cases}$$

where

$$\mathcal{M}\big(n,\ 2k\big) \overset{\text{def}}{=} \min\Big\{ d(G) \big| \forall G = C(N; S), \big|S\big| = 2k \Big\}$$
$$\mathcal{L}(n, 2k) \overset{\text{def}}{=} \max\Big\{ \ell(G) \big| \forall G = C(N; S), \big|S\big| = 2k \Big\}$$

over all connected and undirected circulant networks G of order $n = (2k-1)q + r + 2$ and degree $2k$ where $q \geq 1$, $k \geq 1$, and $r = 0, 1, \ldots, 2k-2$.

In the rest of the paper we shall always assume that G is a circulant network of order n and degree $2k$ such that its jump sizes are in the ordered list S of length $2k$ whose first term is 1.

Theorem 1:

$$\left\lceil \frac{N-2}{2k-1} \right\rceil \leq \mathcal{M}(N, 2k) \leq n - 1$$
$$1 \leq \mathcal{L}(N, 2k) \leq \left\lfloor \frac{(2k-2)N + 2}{2k-1} \right\rfloor$$

Proof: Consider a spanning tree $\mathcal{T}$ of G rooted at u.

Let $A(v) = [v,\ p[v],\ L[v]]$ be the adjacency list of vertex v where $p[v]$ is the parent vertex of v and $L[v]$ is the list of child vertices of v in $\mathcal{T}$.

Since u is the root of $\mathcal{T}$, it follows that $A(u) = [u,\ \text{none},\ L[u]]$ where $L[u] \subseteq S$.

The adjacency lists of the other internal vertices are $[v,\ p[v],\ L[v]]$, where $|L[v]| > 0$, and those of the leaf vertices are $A(z) = \big[z,\ p[z],\ L[z]\big]$, where $L[z]$ is empty.

The number of internal vertices of $\mathcal{T}$ is the maximum if each internal vertex has one child vertex only.

Thus, $d(G) \leq n - 1$ and $1 \leq \ell(G)$. Hence,

$$\mathcal{M}(N, 2k) \leq d(G) \leq N - 1$$
$$1 \leq \ell(G) \leq \mathcal{L}(N, 2k)$$

Since G is $2k$-regular and since the root u has no parent vertex, it follows that $\big|L[u]\big| \leq 2k$, and since each of the other internal vertices of $\mathcal{T}$ has a parent vertex, we have $\big|L[v]\big| \leq 2k - 1$ for all the other internal vertices v of $\mathcal{T}$.

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ISBN: 978-988-19252-0-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Hence, the number of internal vertices of $\mathcal{T}$ can be minimized if the list of child vertices are filled up to its maximum capacity.

The total number of child vertices is $N-1$ since the root has no parent vertex.

If a vertex which is adjacent to the root is first chosen to be its child vertex, then we have $N-2$ child vertices to be distributed to all the $L[v]$'s of the internal vertices, each of which can accomodate up to $2k-1$ child vertices.

Hence, $\left\lceil \dfrac{n-2}{2k-1} \right\rceil \leq d(G)$. This inequality is true for all connected and undirected circulant networks of order n and degree $2k$.

Thus, $\left\lceil \dfrac{n-2}{2k-1} \right\rceil \leq \mathcal{M}(n,2k)$. Consequently, the number of leaf vertices of each of the spanning trees of G is as large as

$n - \left\lceil \dfrac{n-2}{2k-1} \right\rceil = n + \left\lfloor -\dfrac{n-2}{2k-1} \right\rfloor = \left\lfloor \dfrac{(2k-2)n+2}{2k-1} \right\rfloor.$

Hence, $\ell(G) \leq \left\lfloor \dfrac{(2k-2)n+2}{2k-1} \right\rfloor$. This inequality is true also for all connected and undirected circulant networks of order n and degree $2k$.

Thus, $\mathcal{L}(n,2k) \leq \left\lfloor \dfrac{(2k-2)n+2}{2k-1} \right\rfloor$.

Therefore,

$$\left\lceil \frac{n-2}{2k-1} \right\rceil \leq \mathcal{M}(n,2k) \leq n-1$$
$$1 \leq \mathcal{L}(n,2k) \leq \left\lfloor \frac{(2k-2)n+2}{2k-1} \right\rfloor .$$

■

Let $G = C(N;S)$ be a connected and undirected circulant network of order n and degree $2k$ and let $\delta(G)$ be the largest number of vertices between two consecutive jump sizes in G.

Theorem 2: Suppose that $N = (2k-1)q + r + 2$ where $k, q \in \mathbb{Z}^+$, and $r = 0, 1, \ldots, 2k-2$. Then

1) $\delta(G) = q-1$, if $r = 0$ and $S = \{\pm(iq+1)\}$, $\forall i = 0, 1, \ldots, k-1$, or .
2) $\delta(G) = q$, if $r > 0$ and $S = \big\{ \pm s_{1,i}, \pm s_{2,j} \big\}$ where $s_{1,i} = \pm(iq+1)$, $s_{2,j} = \pm((k-r_1+j)q+j+2)$, $\forall i = 0, 1, \ldots, k-r_1-1$, $\forall j = 0, 1, \ldots, r_1-1$, with $r_1 = \left\lfloor \dfrac{r}{2} \right\rfloor$.

Note that the elements of S are computed as residues under modulo N,

Proof:

1) If $r=0$, then $N = (2k-1)q+2$ and if $s_i = iq+1$, for $i = 0, 1, \ldots, k-2$, then

$$s_{i+1} - s_i - 1 = q-1,$$
$$\big(N - s_i\big) - \big(N - s_{i+1}\big) - 1 = q-1,$$
$$\big(N - s_{k-1}\big) - s_{k-1} - 1 = q-1$$

Thus, $\delta(C(N; \pm(iq+1))) = q-1$.

2) For $i = 0, 1, \ldots, k-r_1-1$, and for $j = 0, 1, \ldots, r_1-1$, with $r_1 = \left\lfloor \dfrac{r}{2} \right\rfloor$, let $s_{1,i} = iq+1$ and $s_{2,j} = (k-r_1+j)q+j+2$.

We find the number of vertices between two consecutive jump sizes of $G = C\big(n; \pm(s_{1,i}, s_{2,j})\big)$

$$s_{1,i+1} - s_{1,i} - 1 = q-1,$$
$$\big(N - s_{1,i}\big) - \big(N - s_{1,i+1}\big) - 1 = q-1$$
$$s_{2,j+1} - s_{2,j} - 1 = q$$
$$\big(N - s_{2,j}\big) - \big(N - s_{2,j+1}\big) - 1 = q$$
$$s_{2,0} - s_{1,k-r_1-1} - 1 = q$$
$$\big(N - s_{1,k-r_1-1}\big) - \big(N - s_{2,0}\big) - 1 = q$$

and in the two consecutive jump sizes at the center of S, we have $\big(N - s_{2,r_1-1}\big) - s_{2,r_1-1} - 1 = q + r - 2r_1 - 1$. If r is odd, then $2r_1 = r-1$. If r is even, then $2r_1 = r$. Thus,

$$\big(N - s_{2,r_1-1}\big) - s_{2,r_1-1} - 1 = \begin{cases} q & \text{if } r \text{ is odd, or} \\ q-1 & \text{if } r \text{ is even} \end{cases}$$

Hence, $\delta\big(G\big) = q$.

■

III. An algorithm to construct S, p and L of a spanning tree of G_1 or G_2

Let $\mathcal{A} = \langle A(0),\ A(1),\ \ldots,\ A(n-1) \rangle$ be a sequence of adjacency lists of a tree $\mathcal{T}$ rooted at 0 such that the adjacency list of the root 0 is $\big[0,\ \text{None}, S\big]$ and for $v \neq 0$, we have $A(v) = \big[v,\ p[v],\ L[v]\big]$ where $p[v]$ is the parent vertex of v in $\mathcal{T}$ and $L[v]$ is the list of child vertices of v in $\mathcal{T}$.

Suppose that the respective data structure of p and L are dictionaries where

1) $p = \big\{0 : \text{None},\ 1 : p_1,\ \ ,n-1 : p_{n-1}\big\}$ such that $p[0] = \text{None}$, and $p[v] = p_v$, the parent vertex of v, for each $v = 1, 2, \ldots, N-1$.
2) $L = \big\{0 : S,\ 1 : L_1,\ \ldots,\ \ ,N-1 : L_{N-1}\big\}$ such that $L[0] = S$, and $L[v] = L_v$, the list of child vertices of v, for each $v = 1, 2, \ldots, N-1$.

ALGORITHM 1 Find S, p and L in G_1 or G_2.

Require: $N, k \in \mathbb{Z}$, $n \geq 3$ and $k \geq 1$
Ensure: S, p, and L
if $n - 2*k - 1 < 0$ **then**
  return None
end if
$S \leftarrow [\,]$, $p \leftarrow \{\,\}$, **and** $L \leftarrow \{\,\}$
for $i = 0$ **to** $n-1$ **do**
  $p[i] \leftarrow$ None **and** $L[i] \leftarrow [\,]$
end for
Find $(q, r) = \text{divmod}(N-2,\ 2*k-1)$ {where q is the quotient and r is the remainder when $N-2$ is divided by $2k-1$.}
if $r = 0$ **or** $r = 1$ **then**
  for $i = 0$ **to** $k-1$ **do**
    $u = i*q+1$
    $S \leftarrow S + \big[u, N-u\big]$ {Jump sizes in G_1}
  end for
else $\{r \geq 2\}$
  compute $r_1 = \left\lfloor \dfrac{r}{2} \right\rfloor$
  for $i = 0$ **to** $k - r_1 - 1$ **do**
    $u_1 = i*q+1$

ISBN: 978-988-19252-0-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

    $S \leftarrow S + [u_1,\ N - u_1]$ {jump sizes in G_2}
  end for
  for $j = 0$ **to** $r_1 - 1$ **do**
    $u_2 = (k - r_1 + j) * q + j + 2$
    $S \leftarrow S + [u_2,\ N - u_2]$ {another jump sizes in G_2}
  end for
end if
Sort S in ascending order.
$L[0] \leftarrow S$ {All the jump sizes are child vertices of the root}
for all $v \in S$ **do**
  $p[v] \leftarrow 0$ {0 is the parent vertex of all the jump sizes}
end for
for $i = 0$ **to** $2k - 2$ **do**
  compute $d = S[i+1] - S[i] - 1$
  if $d > 0$ **then**
    for $j = 1$ **to** d **do**
      $v \leftarrow S[i] + j$ $\{v \notin S\}$
      if $v \leq q$ **then**
        $p[v] \leftarrow v - 1$ **and** $L[v-1] \leftarrow L[v-1] + [v]$
      else $\{v > q\}$
        $p[v] \leftarrow j$ **and** $L[j] \leftarrow L[j] + [v]$.
      end if
    end for
  end if
end for
return S, p **and** L.

Theorem 3: Let $N, k \in \mathbb{Z}$ such that $N \geq 3$ and $k \geq 1$ and consider S, p and L generated in Algorithm 1.

Suppose that $\mathcal{A} = \left\langle [v,\ p[v],\ L[v]] \,\middle|\, \forall v \in V\big(G(n;S)\big) \right\rangle$ is the sequence of adjacency lists of the subgraph $\mathcal{T}$ of G. Then

1) $\mathcal{T}$ is a spanning tree of G.
2) The number of internal vertices of $\mathcal{T}$ is q if $r = 0$, or $q + 1$ if $r \geq 1$, and
3) the number of leaf vertices of $\mathcal{T}$ is $N - q$ if $r = 0$, or $N - q - 1$ if $r \geq 1$.

Proof: $\mathcal{T}$ is a spanning subgraph of G since it has all the vertices of G where $G = G_1$ or $G = G_2$.

We shall show that $\mathcal{T}$ is a subtree of G.

Since 0 has no parent vertex in $\mathcal{T}$, it is the root of $\mathcal{T}$.

Let s and t be two consecutive jump sizes in G such that $q = t - s - 1$.

Let v be a nonzero vertex in G. Then v is in one of two closed intervals: $I_1 = [1,\ q]$, or $I_2 = [q+1,\ N-1]$.

1) Suppose $v_1 \in I_1$. Since $s_1 = 1$ and $s_2 = q + 1$, it follows that vertex 1 is the only vertex in I_1 that is a child vertex of the root 0.
   Since $p[v_1] = v_1 - 1$ for all $v_1 \in I_1$, it follows that every vertex in $[1,\ q)$ is a parent vertex in $\mathcal{T}$ and that every vertex in I_1 is connected to the root by a path that passes through vertex 1.
   Since a child vertex has a single parent vertex only, it follows that the path between $v \in I_1$ and 0 is unique for every vertex in I_1.
   a) If $r = 0$, then $\delta\big(G_1\big) = q - 1$. Thus, vertex q has no child vertex in $\mathcal{T}$.
   b) If $r \geq 1$, then $\delta\big(G_1\big) = q$ or $\delta\big(G_2\big) = q$.
      Thus, q has a child vertex in $\mathcal{T}$. Its child vertex is u where $s < u < t$ and $u = s + q$.
   Hence, the number of internal vertices of $\mathcal{T}$ is $q - 1$ if $r = 0$, or it is q, if $r \geq 1$.
2) Suppose $v_2 \in I_2$.
   If $v_2 \in S$, then $p[v_2] = 0$.
   If $v_2 \notin S$, then there exist two consecutive jump sizes s_1 and t_1 such that $s_1 < v_2 < t_1$ and $v_2 = s_1 + u$ for some u in $[1,\ t_1 - s_1 - 1]$.
   Thus, $p[v_2] = u$, since $u \leq t_1 - s_1 - 1 \leq q$.
   This implies that $p[v_2] \in I_1$.
   Hence, the parent vertex for every vertex in I_2 is in I_1.
   This means that $L(v_2)$, which was initialized as the empty list, is always empty.
   Consequently, every vertex in I_2 is a leaf vertex.
   Since each vertex $v_1 \in I_1$ is connected to the root 0 by a path that is unique to v_1 and 0, and since $p[v_2] \in I_1$ for every vertex $v_2 \in I_2$, it follows that v_2 is connected to the root 0 by a unique path in $\mathcal{T}$.
   Hence, the number of leaf vertices of $\mathcal{T}$ is $N - q - 1$ if $r = 0$, or it is $N - q$, if $r \geq 1$.

∎

Let $m(\mathcal{T})$ and $l(\mathcal{T})$ be the number of internal vertices and the number of leaf vertices of a tree $\mathcal{T}$.

Theorem 4: Suppose that G is the circulant network G_1 or G_2 of order N and degree $2k$ where $N \geq 3$ and $k \geq 1$. Then $d(G) = \left\lceil \dfrac{N-2}{2k-1} \right\rceil$ and $\ell(G) = \left\lfloor \dfrac{(2k-2)N+2}{2k-1} \right\rfloor$.

Therefore, $\mathcal{M}(N, 2k) = d(G)$, and $\mathcal{L}(N, 2k) = \ell(G)$.

Proof: By Theorem 3, if $\mathcal{T}$ is the spanning tree of G generated by $\mathcal{A}$ where $G = G_1$ or $G = G_2$ is of order $N = (2k-1)q + r + 2$ and degree $2k$ with $r = 0, 1, \ldots, 2k-2$, $k \geq 1$ and $q \geq 1$, then

$$m(\mathcal{T}) = \begin{cases} q & \text{if } r = 0, \text{ or} \\ q+1 & \text{if } r \geq 1. \end{cases},$$

$$l(\mathcal{T}) = \begin{cases} N - q & \text{if } r = 0, \text{ or} \\ N - q - 1 & \text{if } r \geq 1. \end{cases}$$

Since $\mathcal{M}(N, 2k) \geq \left\lceil \dfrac{N-2}{2k-1} \right\rceil$ and

$$\left\lceil \frac{N-2}{2k-1} \right\rceil = \left\lceil \frac{(2k-1)q + r}{2k-1} \right\rceil$$

$$\left\lceil \frac{N-2}{2k-1} \right\rceil = q + \left\lceil \frac{r}{2k-1} \right\rceil = m(\mathcal{T})$$

Thus, $m(\mathcal{T}) \leq \mathcal{M}(N, 2k)$.

However, by the minimality of $\mathcal{M}(N, 2k)$ and $d(G)$, we have $\mathcal{M}(N, 2k) \leq d(G) \leq m(\mathcal{T})$,

Therefore, $\mathcal{M}(N, 2k) = d(G) = \left\lceil \dfrac{N-2}{2k-1} \right\rceil$.

Also, $\mathcal{L}(N, 2k) \leq \left\lfloor \dfrac{(2k-2)N+2}{2k-1} \right\rfloor$ and

$$\begin{aligned} \left\lfloor \frac{(2k-2)N+2}{2k-1} \right\rfloor &= \left\lfloor \frac{(2k-1)N - N + 2}{2k-1} \right\rfloor \\ &= \left\lfloor \frac{(2k-1)N - (2k-1)q - r}{2k-1} \right\rfloor \\ &= N - q + \left\lfloor \frac{-r}{2k-1} \right\rfloor \end{aligned}$$

$$\left\lfloor \frac{(2k-2)N+2}{2k-1} \right\rfloor = N - q - \left\lceil \frac{r}{2k-1} \right\rceil = l(\mathcal{T})$$

ISBN: 978-988-19252-0-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Thus, $\mathcal{L}(N, 2k) \leq l(\mathcal{T})$.

However, by the maximality of $\mathcal{L}(N, 2k)$ and $\ell(G)$, we have $\mathcal{L}(N, 2k) \geq \ell(G) \geq l(\mathcal{T})$.

Therefore, $\mathcal{L}(N, 2k) = \ell(G) = \left\lfloor \dfrac{(2k-2)N+2}{2k-1} \right\rfloor$. ■

IV. A Spanning Tree of G with Optimal $d(G)$ and $\ell(G)$ but with Lesser Height

The height of a vertex in a tree is the distance of the vertex from the root.

Hence, the height of the root u in a tree is $h(u) = 0$.

The height of v which is a child vertex u is $h(v) = 1$.

The height of a tree $\mathcal{T}$ is defined to be

$$h(\mathcal{T}) \stackrel{\text{def}}{=} \max\{h(v) \big| v \in V(\mathcal{T})\}$$

where $V(\mathcal{T})$ is the vertex set of $\mathcal{T}$.

Hence, the height of the spanning tree $\mathcal{T}$ of G_1 or of G_2 generated by S, p, L of Algorithm 1 is equal to

$$h(\mathcal{T}) = \left\lceil \frac{N-2}{2k-1} \right\rceil .$$

This tree can be shortened by half of its height by modifying the internal vertices in Algorithm 1.

For each $j = 0, 1, \ldots, 2k-2$, let $s_j = S[j]$, $t_j = S[j+1]$, and find $d_j = t_j - s_j - 1$, $c_{j,1} = \left\lceil \dfrac{d_j}{2} \right\rceil$, and $c_{j,2} = \left\lfloor \dfrac{d_j}{2} \right\rfloor$. Then

1) for $i_1 = 1, 2, \ldots, c_{j,1}$,
   $p[s_j + i_1] = i_1$ and append $s_j + i_1$ to $L[i_1]$,
2) for $i_2 = 1, 2, \ldots, c_{j,2}$,
   $p[t_j - i_2] = N - i_2$ and append $t_j - i_2$ to $L[N - i_2]$.

Note that $d_j = c_{j,1} + c_{j,2}$.

Hence, we have the following theorem.

Theorem 5: The spanning tree $\mathcal{T}$ of G_1 or G_2 produced by the modification of the internal vertices of Algorithm 1 as stated above has the following properties:

1) $m(\mathcal{T}) = \left\lceil \dfrac{n-2}{2k-1} \right\rceil$,
2) $l(\mathcal{T}) = \left\lfloor \dfrac{(2k-2)n+2}{2k-1} \right\rfloor$, and
3) $h(\mathcal{T}) = \left\lceil \dfrac{\delta(G)}{2} \right\rceil \leq \dfrac{m(\mathcal{T})}{2}$.

Proof: Let v be a nonzero vertex in $\mathcal{T}$.

If $v \in S$, then $p[v] = 0$. Suppose $v \notin S$.

Then either

- $v = s_j + i_1$ where $i_1 = 1, 2, \ldots, c_{j,1}$ or
- $v = t_j - i_2$ where $i_2 = 1, 2, \ldots, c_{j,2}$.

1) Suppose $v = s_j + i_1$ where $i_1 = 1, 2, \ldots, c_{j,1}$. Then v and i_1 are adjacent vertices in G and $p[v] = i_1$ in $\mathcal{T}$.
   If $v = 1 + i_1$, where $s_0 = 1$ and $v \in \big(1,\ 1 + c_{0,1}\big]$, then $i_1 = v - 1$.
   Consider the vertex $c_{\delta,1} = \left\lceil \dfrac{\delta(G)}{2} \right\rceil$.
   Since $d_j \leq \delta(G)$, it follows that $c_{j,1} \leq c_{\delta,1}$.
   a) Suppose $\delta(G) = q - 1$. Then $d_0 = \delta(G)$.
      Thus, $c_{0,1} = c_{\delta,1}$.
   b) Suppose $\delta(G) = q$. Then $d_0 = \delta(G)$ or $d_0 = \delta(G) - 1$.
      Thus, either $c_{0,1} = c_{\delta,1}$ or $c_{0,1} = c_{\delta,1} - 1$.

   Thus, $c_{\delta,1} \in \big[1,\ 1 + c_{0,1}\big]$. Consequently, $p[c_{\delta,1}] = c_{\delta,1} - 1$.
   Hence, $p[v] = i_1$ in $\mathcal{T}$ where $v = s_j + i_1$ and $i_1 \in \big[1,\ c_{\delta,1}\big] \subseteq \big[1,\ 1 + c_{0,1}\big]$.
   Note that vertex $c_{\delta,1} + 1$ has no child vertex since it is larger than $c_{\delta,1}$.
   Hence, $1, 2, \ldots, c_{\delta,1}$ are the nonzero internal vertices in the closed interval $\big[1,\ 1 + c_{0,1}\big]$ of $\mathcal{T}$.
   These internal vertices form a path between $c_{\delta,1}$ and 0 that passes through 1. This path is unique since a child vertex has a single parent vertex only.
2) Suppose $v = t_j - i_2$ where $i_2 = 1, 2, \ldots, c_{j,2}$. Thus, v and $N - i_2$ are adjacent vertices in G and $p[v] = N - i_2$ in $\mathcal{T}$.
   If $v = t_{2k-1} - i_2 = N - 1 - i_2$ where $1 \leq i_2 \leq c_{2k-2,2}$, then $v + 1 = N - i$. Hence, $p[v] = v + 1$, $v \in [N - 1 - c_{2k-2,2},\ N - 1)$.
   Consider the vertex $N - c_{\delta,2}$.
   Since $\left\lfloor \dfrac{d_j}{2} \right\rfloor \leq \left\lfloor \dfrac{\delta(G)}{2} \right\rfloor$, it follows that $c_{j,2} \leq c_{\delta,2}$.
   Thus, $N - c_{\delta,2} \leq N - c_{j,2}$.
   a) If $\delta(G) = q - 1$, then $d_{2k-2} = \delta(G)$. Thus, $c_{2k-2,2} = c_{\delta,2}$. Hence, $N - c_{\delta,2} = N - c_{2k-2,2}$.
   b) If $\delta(G) = q$, then $d_{2k-2} = \delta(G) - 1$ or $d_{2k-2} = \delta(G)$. Thus, $c_{2k-2,2} = c_{\delta,2}$ or $c_{2k-2,2} = c_{\delta,2} - 1$. Hence, $N - c_{\delta,2} = N - c_{2k-2,2}$ or $N - c_{\delta,2} = N - 1 - c_{2k-2,2}$.
      This implies that $N - c_{\delta,2} \in \big[N - 1 - c_{2k-2,2},\ N - 1\big]$. Thus, $p[N - c_{\delta,2}] = N - c_{\delta,2} + 1$.
      Hence, $p[v] = N - i_2$ where $v = t_j - i_2$, $i_2 = 1, 2, \ldots, c_{j,2}$ such that $N - i_2 \in \big[N - 1 - c_{2k-2,2},\ N - 1\big]$.
      Note that vertex $N - c_{\delta,2} - 1$ has no child vertex in $\mathcal{T}$ since $c_{\delta,2} + 1$ is larger than $c_{\delta,2}$.
      Hence, the nonzero internal vertices of $\mathcal{T}$ in the closed interval $\big[N - 1 - c_{0,2},\ N - 1\big]$ are $N - 1, N - 2, \ldots, N - c_{\delta,2}$.
      This forms a path that passes through $N - 1$ between the root and the internal vertices in this interval.

Thus, the number of nonzero internal vertices are $c_{\delta,1} + c_{\delta,2} = \delta$. Hence, $m(\mathcal{T}) = \delta + 1 = \left\lceil \dfrac{n-2}{2k-1} \right\rceil$.

Consequently, $l(\mathcal{T}) = \left\lfloor \dfrac{(2k-2)n+2}{2k-1} \right\rfloor$. The height of the tree is equal to $h(\mathcal{T}) = c_{\delta,1} \leq \dfrac{m(\mathcal{T})}{2}$.

■

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ISBN: 978-988-19252-0-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)