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Color Models as Tools in Teaching Mathematics

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Abstract: In this paper we discuss various situations how color models and patterns can be used to simplify the study of abstract mathematics and serve as tools in understanding mathematical ideas. We illustrate the realization of such models through the development of advanced computer technology. In particular, we present how a computer algebra software such as *Mathematica*, or a dynamic geometry environment, can be utilized to facilitate the study of transformation geometry and group theory.

1. Introduction

In the mathematics classroom, a lecture or presentation can be made more interesting and can captivate students' attention with the use of color models and colored patterns. More importantly, these models can be used as tools in teaching and learning algebraic and geometric theories; and make possible the visualization of abstract concepts and the development of critical thinking.

In many universities and colleges, abstract algebra is one of the courses where mathematics majors, future mathematics teachers or teachers studying graduate courses in mathematics have difficulty, either in dealing with the content or in terms of developing a positive attitude to the subject matter. The underlying concepts are difficult to grasp because of the high level of abstraction and complexities involved, and students also have to understand and write proofs.

The purpose of this note is to discuss the use of color models and colored patterns in teaching topics in abstract algebra such as group theory. The use of technology in the classroom makes accessible these models via dynamic geometry software, computer algebra systems or interactive software from the internet. Not only can students work in an exploratory environment wherein they can make and test conjectures, but also more importantly, with technology, they are allowed to think visually and geometrically. Asking students to validate proofs using specific technology based activities may not only help students develop the ability to validate proofs, but also may help increase their own proof writing abilities. As the students attempt to familiarize themselves with the concepts by manipulating these models and try to develop new ideas, they become less intimidated with the abstraction and gain positive outlook to the course.

2. Colored Cayley tables and other models from *Mathematica*

Arthur Cayley in 1854 introduced a procedure for defining a finite group G by listing its elements in the form of a multiplication table which was later known as a *Cayley table* representing G. In this section, we give some colored Cayley tables and other models, which can serve as tools for the study of group and subgroup properties, cosets; concepts on homomorphisms and isomorphisms, rings and fields. In our work, we generate the models through the add-on Abstract Algebra package called by the computer algebra software *Mathematica* [11].

A. Group and Subgroup Properties.

We first show models used to study some properties of a group, its subgroup and elements.

As a first example, we focus our attention to the cyclic group Z_8 consisting of 8 elements, under $+_8$ the operation addition modulo 8. Consider the Cayley table corresponding to Z_8 given in Figure 1. In the table a different color is used for each element of a group. The entries in the table corresponding to the elements are colored and labeled accordingly. Just by looking at the colors in the table one can verify easily the group properties and characteristics of Z_8. Notice that the group identity will have to be the element 0, (represented by color yellow), since it is the column under this element that will give the same sequence of colors yellow (0), pink(1), orange(2), ... , and brown(7), indicating that upon application of yellow to every group element, the identity of each element has been preserved. The inverse of each group element can also be identified easily. Consider for example the element 6, represented by the color green. The inverse is clearly the element 2, colored orange since upon application of orange to green, the result is yellow, which represents the identity of the group.

To understand the orders of the elements of Z_8, a Cayley table colored in a different way, such as the one shown in Figure 2 may be used. The palette found at the bottom of the table indicates the color corresponding to a group element raised to the *nth* power. Properties pertaining to an element $g \in Z_8$ (listed in column 1), can be verified in the corresponding row of g. For instance, if one would want to determine the result of g^2, one would simply look at the group element in the row of g that is given the color orange, e.g., 5 added twice gives 2, as shown in row 6.

Figure 1: Cyclic group Z_8 **Figure 2**: Orders of elements of Z_8 **Figure 3**: Subgroup <6> of Z_8
(Outputs produced using *Mathematica* [11])

Now to determine the order of g, one would look at the color assigned to 0 in the corresponding row of the table. The order of 4 can be verified to be 2 since 4 added twice gives orange, which is the color corresponding to 0 in the fifth row. To find the generators of the group, once can inspect those rows with 8 complete colors, indicating that all elements of the group have been generated by an element. For the group Z_8, these will be the elements 1, 3, 5 and 7. For a non-generator of Z_8, say 2, one can also pick out the elements of the group <2> generated by 2, namely 2,4, 6 and 0 which are those that come with colors in the third row. Note that color tables on the orders of the elements of a group, such as that presented in Figure 2 can also help illustrate differences between cyclic and non-cyclic groups, where students can verify that for a cyclic group, there are elements that generate the whole group.

Properties pertaining to subgroups can also be verified from similar colored Cayley tables of subgroups. An example of a Cayley table of the subgroup <6> of Z_8 is shown in Figure 3.

In general, a cyclic group Z_n may also be viewed through a clock model, where we think of the elements as numbers on an n hour clock where the nth element is viewed as being equivalent to zero. Addition of two numbers under $+_n$ in Z_n will be treated like adding numbers on the clock. For the cyclic group Z_8, we view the last element 8 as being equivalent to zero.

Properties on subgroups can be derived from the clock model: In Figure 4 we present the clock model for the subgroup <6> generated by the element 6. One can visualize the path traced on the clock model as the subgroup is generated; that is 6 generates the subgroup {6, 4, 2, 0} of 4 elements. The inverse of 6, which is 2, also generates the same subgroup, as shown in Figure 5. Students can arrive at the generalization that an element x and its inverse, n- x, generates the same subgroup, that is, $< x > = < n - x >$. Figure 6 shows the model of the group generated by 3, which happens to be Z_8; the path forms an 8-star.

Activities can be created where the students explore the generators of Z_n - these generators are elements a^k such that $(k,n) = 1$. The clock model clearly describes the fact that the group is cyclic since after n applications of the generator under $+_n$,we complete a cycle and start from zero again. Moreover, students can also visualize the construction of a star of n sides and points from a generator a^k of Z_n where $\gcd(n, k) =1$ and $1< k < n - 1$. The visual representation of the cyclic group Z_n given by the construction of the n-point star is an interesting motivational tool that can be used in the teaching of such abstract concepts [1,2].
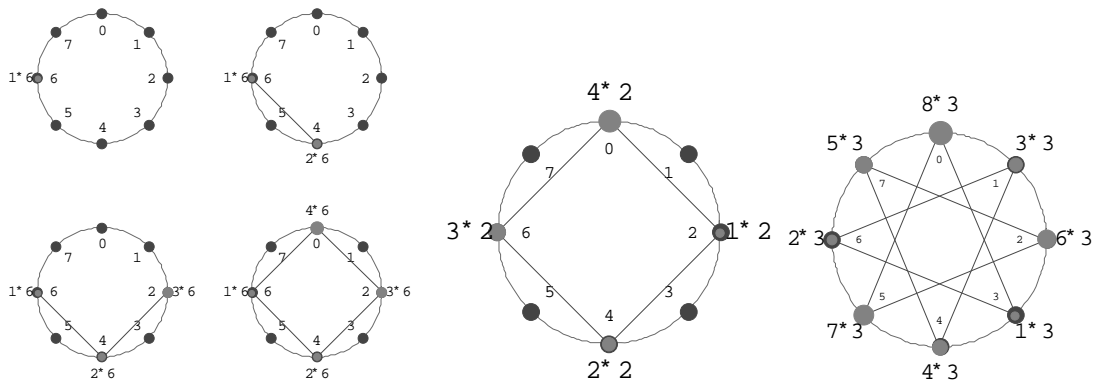

Figure 4: Subgroup <6> of Z_8 **Figure 5**: Subgroup <2> of Z_8 **Figure 6**: Subgroup <6> of Z_8
(Outputs produced using *Mathematica* [11])

Models corresponding to elements of the dihedral group D_n, as symmetries of a regular *n-gon* can also be generated by *Mathematica*. For instance, the effect of an element of the group D_3 as a symmetry of the equilateral triangle is visualized from the screen output presented in Figure 7. The vertical reflection $x \in D_3$ fixes the vertex 3 and interchanges vertices 1 and 2. From the model, this element can also be understood as a permutation element of the symmetric group S_3. The element x is the two cycle $(12) \in S_3$. Using similar models, it is possible to visualize group elements as symmetry elements. For example, we can also exhibit models corresponding to the rotational symmetries of the tetrahedron, which constitute the alternating group A_4.
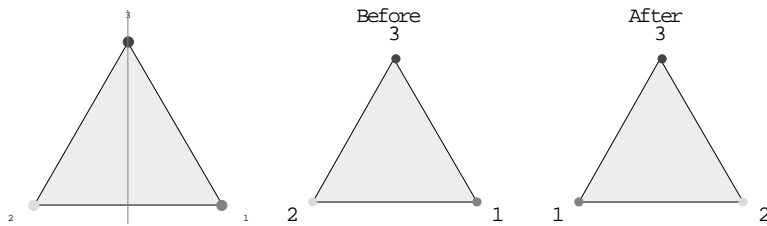
Figure 7: The effect of the vertical reflection *x* on the equilateral triangle
(Output produced using *Mathematica* [11])

B. Cosets, Isomorphisms and Homomorphisms

Colored Cayley tables may help explain the construction of the quotient group *G/H* of left cosets of a given normal subgroup *H* of a group *G* as well as illustrate the concept of isomorphism between two given groups.

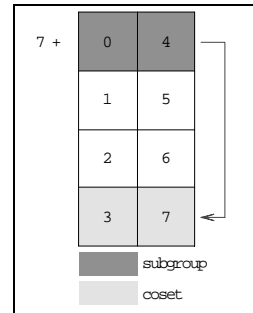**Figure 8:** Cosets of *H*= {0,4} in Z_8 **Figure 9**: Coset 7 + *H*
(Outputs produced using *Mathematica* [11])

To view examples of these colored models, let us again look at the cyclic group Z_8. Consider a particular subgroup *H* = {0,4} of *G*. The set of left cosets of *H* in *G* are the following, namely, *H*, 1+*H*, 2+*H* and 3 + *H*, and these form the group *G/H*. The structure of this group can be understood more clearly through the colored Cayley table shown in Figure 8. Each left coset is given one color in the table: yellow, pink, orange and green. Thus one can easily determine the elements of each coset, since all elements belonging to one coset are given the same color. Also using the same procedure discussed in Section 2.A, one can also verify the identity and the inverse of each element of *G/H*. A detailed computation of how a particular coset is obtained can also be viewed through a different table. For example, the computation of how the set 7+*H* has been calculated can be visualized in Figure 9.

The group *G/H* is a finite group of 4 elements. To verify the structure of *G/H*, we can generate colored Cayley tables of the two finite groups of order 4, namely, the Klein 4 group and the cyclic group Z_4, and compare these with the colored Cayley table of *G/H*. From the tables given in Figure 10, it can be deduced quickly by observing the color schemes that *G/H* and Z_4 have indeed the same group structure, or we can say that *G/H* is isomorphic to Z_4. Note that *G/H*, and Z_4 are also isomorphic to the subgroup <6> of Z_8. (Compare Figures 10 and 3).

Z[8] x + y

x\y	0	4	1	5	2	6	3	7
0	0	4	1	5	2	6	3	7
4	4	0	5	1	6	2	7	3
1	1	5	2	6	3	7	4	0
5	5	1	6	2	7	3	0	4
2	2	6	3	7	4	0	5	1
6	6	2	7	3	0	4	1	5
3	3	7	4	0	5	1	6	2
7	7	3	0	4	1	5	2	6

Z[4] x + y

x\y	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Klein4 x * y

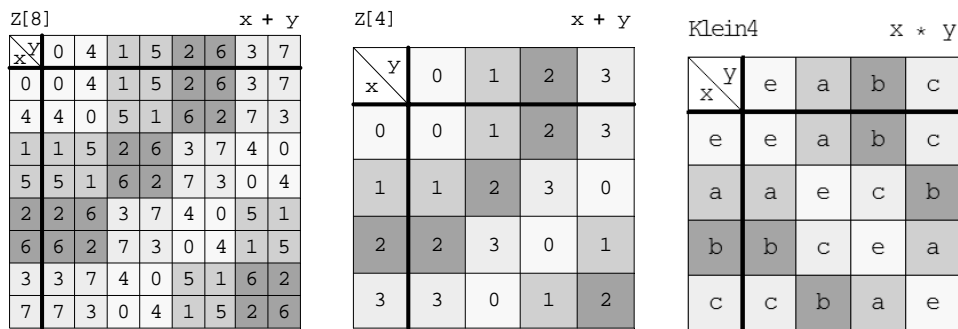| x\y	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Figure 10: Cayley tables of $Z_8/\{0,4\}$, Z_4, Klein 4
(Outputs produced using *Mathematica* [11])

The last part of this section is devoted to an example of how particular color models help us determine homomorphisms from a cyclic group to another cyclic group.

Given the cyclic group Z_4 and Z_{12}, we define the map g from Z_4 to Z_{12} that sends the element 1 in Z_4 to the element 6 in Z_{12}, visualized in Figure 11. This map g is a homomorphism and we can confirm this via the corresponding colored table. The table gives us information whether a given map h is a homomorphism in the following manner. The table entry corresponding to the computation $a*b$ in the domain of the homomorphism is colored if and only if the pair $\{a,b\}$ is preserved by the homomorphism, that is, $h(a*b) = h(a)*h(b)$. Notice that the table corresponding to g has all its entries colored, thus g is a homomorphism.
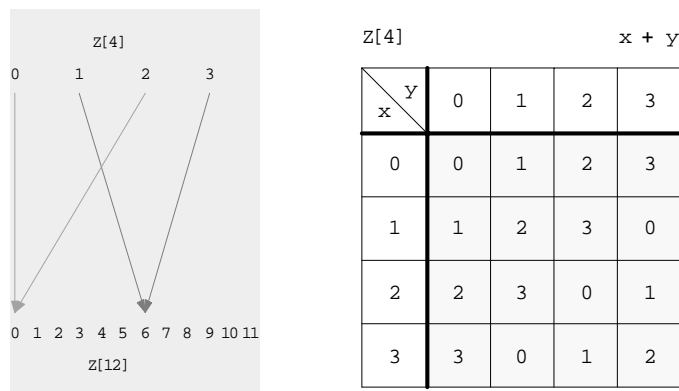
Z[4]

x\y	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

x + y

(flowchart: Z[4] elements 0 1 2 3 mapping to Z[12] elements 0 1 2 3 4 5 6 7 8 9 10 11)

Figure 11: Visualization of homomorphism g from Z_4 to Z_{12}
(Outputs produced using *Mathematica* [11])

Now, let us define the map f from Z_4 to Z_{12} that takes the element 1 in Z_4 to the element 4 in Z_{12}, This map can be visualized in Figure 12. Using the colored table that corresponds to f, we can verify that this map is not a homomorphism, since not all the entries in the table is colored. For example, note that the entry corresponding to 2+3 is not colored, which suggests that $f(2+3) \neq f(2) +f(3)$. We can understand this better by looking at the companion colored flowchart. We have $f(2+3) = f(1) =4$; whereas $f(2) +f(3) = 0+ 8 =8$.
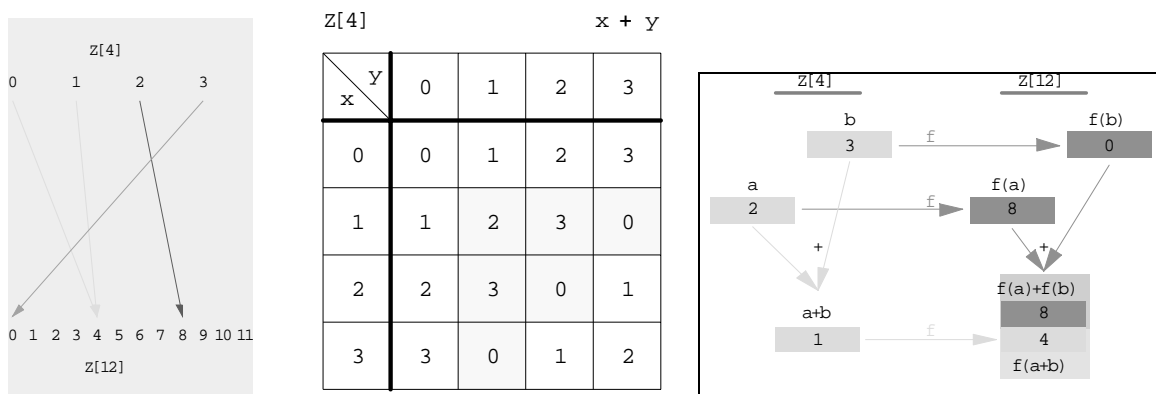
Figure 12: Visualization of homomorphism *f* from Z_4 to Z_{12}
(Outputs produced using *Mathematica* [11])

C. Rings and Fields

In Figures 13 and 14, we present as examples the colored tables for the respective rings Z_5 and Z_8 (under addition, multiplication mod 5 and 8). Students can compare the structures of both rings from the tables. The ring Z_5 is a field, whereas Z_8 is not. A close inspection of the multiplication table for Z_8 will make evident that elements corresponding to rows without "1" (pink color), which is the multiplicative identity, do not have multiplicative inverses. By exploring several tables of representative rings, students can answer questions like when is the ring Z_n a field? Then students attempt to formulate a proof.

Figure 13: Tables for ring Z_5 **Figure 14**: Tables for ring Z_8
(Outputs produced using *Mathematica* [18])

3. Color Models from Interactive Software

There are also color models for teaching and visualizing abstract algebra that can be generated through the interactive software *Group Explorer* [9], which is an open-source software that can be easily downloaded from the internet (http://groupexplorer.sourceforge.net/). These models not only come in the form of group tables, but also in terms of Cayley diagrams, cycle graphs and objects of symmetry. There is a wide variety of finite groups to explore- cyclic and dihedral groups, the quaternions, etc. There is room for flexibility in manipulating the models. This feature of the software gives the students more opportunities for exploration and investigation. For instance, for the group tables, one can change relevant properties such as order of rows and columns, color scheme, and one can separate the group elements into cosets. Lattice of subgroups can also be

generated, showed in terms of tables. Moreover, a symmetry object maybe clicked and dragged, so the student can view and rotate it in three-dimensional space.

e	r	r^2	r^3	f	rf	r^2f	fr
r	r^2	r^3	e	rf	r^2f	fr	f
r^2	r^3	e	r	r^2f	fr	f	rf
r^3	e	r	r^2	fr	f	rf	r^2f
f	fr	r^2f	rf	e	r^3	r^2	r
rf	f	fr	r^2f	r	e	r^3	r^2
r^2f	rf	f	fr	r^2	r	e	r^3
fr	r^2f	rf	f	r^3	r^2	r	e

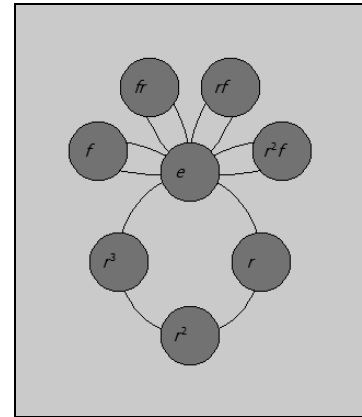| e	f	r	fr	r^2	r^2f	r^3	rf
f	e	fr	r	r^2f	r^2	rf	r^3
r	rf	r^2	f	r^3	fr	e	r^2f
fr	r^3	r^2f	e	rf	r	f	r^2
r^2	r^2f	r^3	rf	e	f	r	fr
r^2f	r^2	rf	r^3	f	e	fr	r
r^3	fr	e	r^2f	r	rf	r^2	f
rf	r	f	r^2	fr	r^3	r^2f	e

Figure 15: Dihedral group D_4 **Figure 16**: A coset table for D_4 **Figure 17**: Cycle graph for D_4
(Outputs produced using *Group explorer* [9])

As an example, we consider the dihedral group of 8 elements, $D_4 = <r, f> = \{e, r, r^2, r^3, f, rf, r^2f, fr\}$. Figure 15 shows the Cayley table for D_4 with each element given a particular color. On the other hand, Figure 16 illustrates the partition of D_4 into cosets of the subgroup $S = \{e, f\}$ of D_4. An example of a cycle graph output, generated by *Group explorer* is shown in Figure 17 for D_4 - there is evidence of a one 4-cycle, and four 2-cycles in the graph. A cycle graph gives an illustration of the cycles of a group, that is, the orbits of elements and how the cycles connect. Thus D_4 has four cyclic subgroups of order 2, and one cyclic subgroup of order 4. Recall that a cyclic group can be generated by one element, and the group consists of entirely the orbit of that element.

A Cayley diagram is a directed graph with a node corresponding to each element of the group, and a type of arrow for each group generator, which is color coded by *Group explorer.* Cayley diagrams show the actions of the generators on all the group elements. The diagram may also be clicked and dragged so various views are possible. Nodes can be given colors to highlight a particular subgroup (see Figure 18). Figures 18 and 19, shows, respectively, screen captures of the Cayley diagram and object of symmetry (square) of D_4. From the Cayley diagram of D_4, students can deduce the facts that the group is generated by two elements; and that the group is non-abelian by exploring the orders of the arrows. This feature of *Group explorer*, that of generating Cayley diagrams, come very handy in the classroom when presenting links of group theory to graph theory, and applying these diagrams to represent real world situations, such as networking problems.
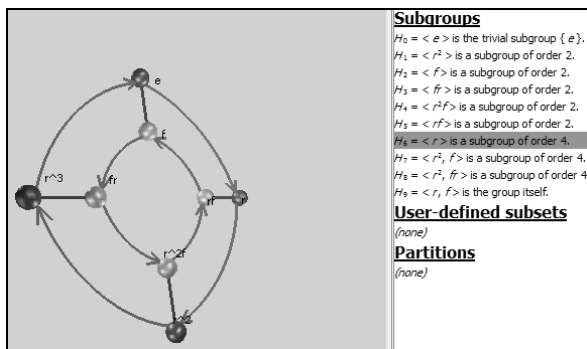

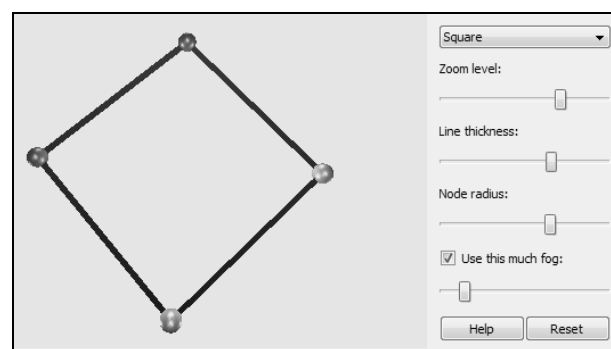**Figure 18**: Cayley diagram of D_4 **Figure 19**: Object of symmetry of D_4
(Outputs produced using *Group explorer* [9])

There are pre-constructed activities that may be used in teaching abstract algebra, one source is the Wolfram Demonstrations project site [14] which features a collection of animated demonstrations involving a wide range of mathematical topics. One activity involves the use of colored models to illustrate the concept of group actions, orbits and stabilizers. For example in Figure 16, we present a still image of a sample demonstration. In the given activity, the idea is to visualize the resulting action of a group on the set of colors of the squares. For a particular coloring displayed on the left window, the images under the action of a group element on the coloring are displayed on the right, as are the orbits of colors and stabilizer. The group can be a choice of a dihedral group of order 8 or a cyclic group of order 4, acting on either two, three, or four colors.

By analyzing the resulting colored patterns after varying the groups and the color assignments to the squares, the concepts on orbits and group actions are better understood. The use of technology in this activity allows a geometric representation of the abstract concepts through the use of colored patterns and facilitates the exploration process that leads to a deeper understanding of the concept. The fact that technology can generate these images with ease allows the students to observe group properties through the colored patterns that result from the group action, and give concrete generalizations. By also giving representations to the colors (e.g. a color represents an atom or molecule in a given crystal) the students will appreciate the connections of the concepts to real world situations. In future studies this will help the students integrate such mathematical concepts with other fields of study.
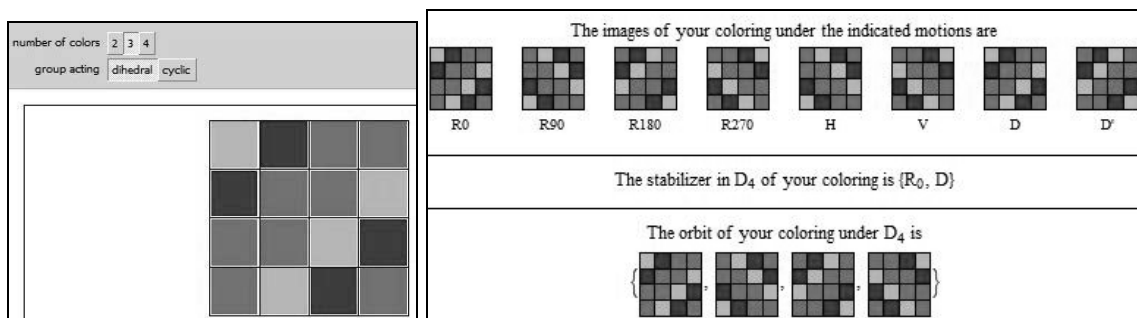

Figure 20: Brodie's *Orbits and Stabilizers of Groups Acting on Colorings of* 4x4 *Checkerboards* (Still image from Wolfram Demonstration's Project [12])

4. Dynamic Geometry Activities

In this part of the paper, we highlight some investigations which can be carried out in a dynamic geometry environment to understand group theoretic concepts, in particular using colorings of left cosets and right cosets of a given subgroup.
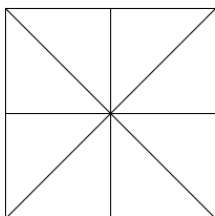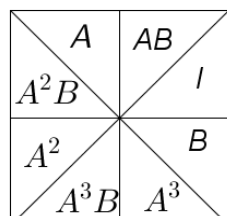

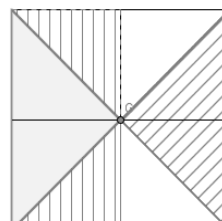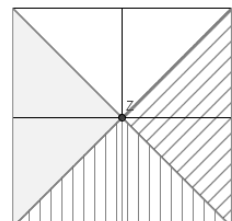**Figure 21**: pattern P **Figure 22**: labeling of P **Figure 23**: right coset coloring   **Figure 24**: left coset coloring

(Construction produced using Geogebra [8])

We assume we have a pattern on the plane whose symmetry group is either a dihedral group, a cyclic group, a frieze group or a plane crystallographic group. For example the uncolored pattern P in Figure 21 has as symmetry group G the dihedral group of eight elements generated by a counterclockwise 90° rotation A about the center of the square and a mirror reflection B in the horizontal line through the center of the square. By labeling one of the triangles as I, corresponding to the identity element of G, we may label each of the eight triangles by the elements of G which sends the triangle I to the specified triangle. This sets up a one-to-one correspondence between the set of eight triangles and the set G, as shown in Figure 22. A colored pattern is arrived at by assigning a color to each element of G, and thus to each of the eight triangles. For example, the colored pattern $P*$ in Figure 23 is a coloring using right cosets $H = \{I,B\}$(pink hatch), $HA = \{A, A^3B\}$ (green hatch), $HA^2 = \{A^2, A^2B\}$(blue) and $HA^3 = \{A^3, AB\}$ (white).

Some questions that may be asked are: what is the symmetry group of the colored pattern? If a mirror reflection AB is performed where will each of the colors go? Which symmetries effect a permutation of the colors? What is the normalizer of H in G?
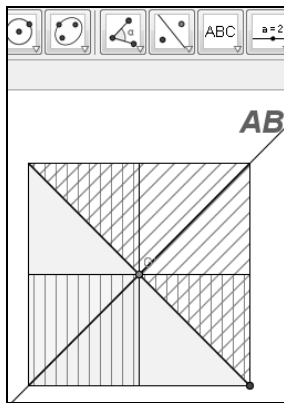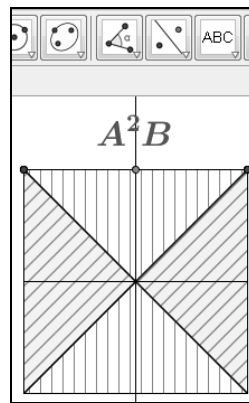

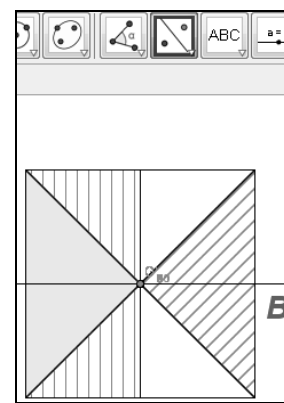**Figure 25**: Applying AB **Figure 26**:Applying A^2B **Figure 27:** Applying B
(Images produced using Geogebra [8])

Working within a dynamic geometry environment, investigations may be carried out to answer these questions. For example, applying the reflection AB to $P*$, we get an inconsistent overlap of colors, as shown in the screen capture given in Figure 25. Pink portions overlap with green as well as white. On the other hand, when applying the reflection A^2B, we obtain a consistent overlap of colors, suggesting that A^2B effect a permutation of the colors; as shown in Figure 26. It can be verified that the subgroup of elements of G that effect a permutation of the colors is $\{I, A^2B, B, A^2\}$. For a coloring using right cosets of S in G, the subgroup of elements which permute the colors is the normalizer of S in G, $N_G(S)$. Thus, from the activity, it can be concluded that $N_G(H) = \{I, A^2B, B, A^2\}$. The elements of $N_G(H)$ that fix the colors is the symmetry group of the colored pattern. For example applying the reflection B sends the colored pattern $P*$ to itself (Figure 27). Thus the symmetry group of the colored pattern is $\{I,B\}$.

Now for the subgroup $H = \{I, B\}$, the four left cosets are H, AH, A^2H and A^3H. A coloring of P using these left cosets is shown in Figure 24. It can be checked by application of each symmetry $g \in G$ to the colored pattern, that each element of G effects a permutation of the colors. A question that may be asked is what elements of G fix each color? For example, the identity I and the reflection B fix the blue colors given by the coset A^2H. Note that the stabilizer of a color gH, or the subgroup fixing color gH is the conjugate of H by g, gHg^{-1}. Thus $A^2HA^2 = \{I,B\}$. Visually, this gives a way of determining conjugate subgroups using colors, or through color symmetry.

5. Conclusion

Here we have presented an approach in teaching abstract algebra, with the aid of color models and patterns. The approach addresses studying the concepts in a more visual and geometric way. Moreover, the approach is facilitated and made easier with the use of either the computer algebra software *Mathematica*, dynamic geometry technology and interactive software; or a combination of these technological tools.

Rather than following a definition-theorem-proof-example format fare usual in teaching abstract algebra, our approach in teaching the course was to first engaged the students in activities with concrete examples; working with two or three partners. In the previous sections we have presented some of the models and the activities that we have used in our classes. These were either done in class or as homework to be carried out outside class hours. Then drawing from the students' observations and conjectures, we formulate the definition or attempt to prove the generalization. The reflective discussions that followed the activities were valuable; these allowed the students to help concretize ideas which were later formalized into the definitions, theorems and were tools when they tried to write proofs.

An additional helpful supplement to teaching abstract algebra is the interactive site created by Joseph Gallian [15]. The site is a supplement to [6], but can work hand-in-hand with other textbooks. The technology exercises are designed to help formulate and test conjectures.

There is still a lot to be learned in terms of strategies of teaching abstract algebra with color models through technology tools, and a direction to be pursued for future work would be to study the usefulness of these approaches on student learning.

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[9] Group Explorer 2.2, developed by Nathan Carter, 2009. http://groupexplorer.sourceforge.net/
[10] Java, developed by Sun Microsystems. http://java.com/en/
[11] Mathematica, developed by Wolfram Research. http://www.wolfram.com/
[12] Orbits and Stabilizers of Groups Acting on Colorings of 4x4 Chessboards, Wolfram Demonstration Project, contributed by Marc Brodie. http://demonstrations.wolfram.com/OrbitsAndStabilizersOfGroupsActingOnColoringsOf4x 4Chessboard/
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[15] Software Supplement to Contemporary Abstract Algebra by Gallian http://www.d.umn.edu/~jgallian/msproject06/project_xukai.html